Some Example Solved by The Double $(\phi'/\phi, 1/\phi)$-Expansion Method

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1. The Double $(\phi'/\phi, 1/\phi)$-Expansion Method

Consider the second-order equation.

$$\phi''(\xi)+\delta\phi(\xi)=q,$$

(1.1) where δ and q are subjective parameters. We can consider the relationship in the following way:

$$\phi=\phi'/\phi \text{ and } \psi=1/\phi.$$

(1.2) If differentiate equation (1.2) with respect to ξ, the result obtained is as follows:

$$\phi'(\xi)=q\psi(\xi)-\delta-\phi^2(\xi), \psi'(\xi)=-\phi(\xi)\psi(\xi).$$

(1.3) Depending on the value of δ, equation (1.1) have three potential generic solutions.

For type 1 ($\delta<0$): When negative score is allotted to δ, the general solution of Eq. (1.1) will comprise hyperbolic functions in the following form:

$$\phi(\xi)=B_1\sinh(\sqrt{-\delta}\,\xi)+B_2\cosh(\sqrt{-\delta}\,\xi)+\frac{q}{\delta}$$

(1.4) together with

$$\psi(\xi)^2=\frac{\delta(\phi^2(\xi)-2q\psi(\xi)+\delta)}{\delta^2 S+q^2},$$

(1.5) whereas B_1, B_2 are arbitrary constants and $S=B_1^2-B_2^2$.

For type 2 ($\delta>0$): When positive score is allotted to δ, the general solution of Eq. (1.1) will comprise trigonometric functions in the following form:

$$\phi(\xi)=B_1\sin(\sqrt{\delta}\,\xi)+B_2\cos(\sqrt{\delta}\,\xi)+\frac{q}{\delta}$$

(1.6) along with

$$\psi(\xi)^2=\frac{\delta(\phi^2(\xi)-2q\psi(\xi)+\delta)}{\delta^2 S-q^2},$$

(1.7) wherein B_1, B_2 are arbitrary constants and $S=B_1^2+B_2^2$.

Type 3 ($\delta=0$): When δ is equal to zero, the general solution of Eq. (1.1) takes the form of a rational function as follows:

$$\phi(\xi)=\frac{q}{2}\xi^2+B_1\xi+B_2,$$

(1.8) with

$$\psi(\xi)^2=\frac{(\phi^2(\xi)-2q\psi(\xi))}{(B_1^2-2qB_2)},$$

(1.9) where B_1, B_2 are arbitrary constants.

Consider a mathematical model that represents a complex issue and incorporates fractional-order differentiation of a function $\psi=\psi(x,t)$:

$$F(\psi, D_t^{\alpha}\psi, D_t^{2\alpha}\psi, D_x^{2\alpha}\psi, D_{xxt}^{3\alpha}\psi, D_{xxx}^{3\alpha}\psi, ...)=0, 0<\alpha<1,$$

(1.10) wherein F is a polynomial that comprises both ψ and its fractional derivatives. Next, we will outline the fundamental stages of the $(\phi'/\phi, 1/\phi)$-expansion approach for two variables (Odabasi 2020):

1st Step: The following fractional wave transformation:

$$\psi(x,t)=U(\xi), \xi=\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}-\frac{\rho}{v}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}.$$

(1.11) where α represents fractional-order derivative, ξ stipulates wave variable, v signifies wave number, and ρ specifies wave velocity. If the equation (1.10) undergoes the wave transformation (1.11), it results the following nonlinear equation:

$$M(U, U', U'', U''', ...)=0,$$

(1.12) where M represents a polynomial relating U and its associated derivatives.

2nd Step: Following this approach, a polynomial solution depending on the variables ϕ and ψ of Eq. (1.12) can be expressed as:

$$U(\xi)=\sum_{i=0}^{N} a_i\phi^i+\sum_{i=1}^{N} b_i\phi^{i-1}\psi, \phi=\phi'/\phi, \psi=1/\phi,$$

(1.13) where $a_i(i=0,1,...,N)$ and $b_i(i=1,...,N)$ are constants.

3rd Step: The integer N can be determined by following the balancing principle between the nonlinear terms and the derivatives of the highest order in Eq. (1.13).

4th Step: Setting the trial solution from equation (1.13) into equation (1.12), a new equation can be obtained that involves ϕ, ψ and their derivatives. By putting the specific values of ϕ', ψ', and ψ^2, the higher-order terms of ψ and its derivatives in the resulting new equation can be removed. Now, we equate the coefficients of $\phi^i\psi^j(i=0,1,\ldots,N, j=0,1)$ to zero on both sides of the equation, we obtain a set of algebraic equations.

5th Step: At the end, unraveling this algebraic system of equations leads to the generation of several sets of solutions. These sets of solutions are then utilized in the trial solution (1.13) to derive the necessary analytical traveling wave solutions.

The double $(\phi'/\phi, 1/\phi)$-expansion method is a more versatile and powerful extension of the original (ϕ'/ϕ)-expansion method. While the original method is simpler and suitable for basic FNLEEs, the double method addresses its limitations which offer a broader range of solutions and better handling

the complexity of fractional-order systems. The original (ϕ'/ϕ)-expansion method is implemented in standard FNLEEs and is often used as a baseline for studying fractional differential equations in mathematics and physics. In contrast, the double $(\phi'/\phi, 1/\phi)$-expansion method is widely applied to fractional dynamics, optical fiber communication, ion-acoustic models, and fluid mechanics due to its ability to produce precise and diverse solutions. It serves as a valuable and powerful tool for studying FNLEEs in real-world applications.

2. Applications

Analytical solutions have several advantages over numerical solutions in various contexts. The objective of this part is to use the $(\phi'/\phi, 1/\phi)$-expansion method to obtain a stable broad-ranging general closed-form solutions to the space-time fractional NNV equation and the space-time fractional nonlinear Schrödinger Kerr law equation, the space-time fractional nonlinear Schrödinger unstable equation, the space-time fractional Schrödinger-Hirota equation, the space-time fractional Biswas-Arshed equation from which existing solutions and standard closed-form wave solutions can be extracted.

2.1. The fractional (2+1)-dimensional Nizhnik-Novikov-Veselov equation

The (2+1)-dimensional fractional equations are particularly significant because they describe physical phenomena that occur in two spatial dimensions and one temporal dimension. For instance, the fractional (2+1)-dimensional NNV equation authors (Zhang et al., 2022; Duarte et al., 2017; Fong, 2015; Wang and Dai, 2015; Williams, 2017; Chen and Wang, 2020; Zhao and Zhang, 2023; Abdalla et al., 2018; Wang, and Liu, 2022) describes the behavior of particles in a two-dimensional space-time environment under the influence of a potential energy field. It is widely used in quantum mechanics and condensed matter physics. This makes fractional integrable equations a valuable tool for physicists and engineers working in fields where non-integer order derivatives play a role. Overall, fractional integrable equations, including the (2+1)-dimensional equations, provide a powerful mathematical framework for modeling and understanding a wide range of physical phenomena, and they have important applications in various fields of physics and engineering. The (2+1)-dimensional time-fractional Nizhnik-Novikov-Veselov equation is defined as:

$$D_t^{\alpha}\psi = a\psi_{xxx} + b\psi_{yyy} - 3a\varphi_x\psi - 3a\varphi\psi_x - 3b\omega_y\beta - 3b\omega\psi_y,$$

(2.1.1)

$$\psi_x = \varphi_y, \psi_y = \omega_x.$$

where a, b are nonzero constants and ψ, φ and ω are the functions of x, y, t. A nonlinear geometric optics model was used to develop the non-dimensional Nizhnik-Novikov-Veselov equation. Consider the wave transformation

$$\xi = x + y - \frac{k}{a}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha},$$

(2.1.2) where $\psi(x, y, t) = U(\xi), \varphi(x, y, t) = V(\xi), \omega(x, y, t) = W(\xi)$, (2.1.3) where k is a constant and $0 < \alpha \leq 1$. Using the wave transformation (2.1.2), Eq. (2.1.1) is remodeled as follows:

$$(a+b)U'''-3a(VU)'-3b(WU)'-kU'=0,$$

(2.1.4)

$$U'=V',U'=W'.$$

(2.1.5) Integrating equations (2.1.4) and (2.1.5) and simplifying, it is attained

$$(a+b)U''-3a(VU)-3b(WU)-kU+e=0,$$

(2.1.6)

$$V=U+c,W=U+d.$$

(2.1.7) Applying equation (2.1.7) into equation (2.1.6), we attain the subsequent form

$$(a+b)U''-3(a+b)U^2+(3ac+3bd+k)U+e=0.$$

(2.1.8) By using balancing principle, we acquired $N=2$. Therefore, the trial solution of equation (2.1.8) can be considered as

$$U(\xi)=a_0+a_1\varphi(\xi)+a_2\varphi^2(\xi)+b_1\psi(\xi)+b_2\varphi(\xi)\psi(\xi),$$

(2.1.9) where a_0,a_1,a_2,b_1 are b_2 constants.

Case 1: If $\delta<0$. Putting solution (2.1.9) into (2.1.8), using (1.3) and (1.5) the left hand side becomes a polynomial in φ and ψ. Setting the coefficients of the polynomial to zero; a group of algebraic equations are obtained. Solving these algebraic equations for a_0,a_1,a_2,b_1,b_2 and k, using the Mathematica program, we attain two sets of values provided below:

$$a_0=\delta,a_1=0,a_2=1,b_1=-q,b_2=\sqrt{\frac{-(\delta^2S+q^2)}{\delta}},$$

$$k=a\delta+b\delta-3ac-3bd,$$

(2.1.10) and

$$a_0=\frac{2}{3}\delta,a_1=0,a_2=1,b_1=-q,b_2=\sqrt{\frac{-(\delta^2S+q^2)}{\delta}},$$

$$k=-a\delta-b\delta-3ac-3bd.$$

(2.1.11) Using the values of the parameters gathered in (2.1.10) into solution (2.1.9), we find the solution

$$U(\xi)=\delta-\delta\frac{\left\{B_1\cosh(\xi\sqrt{-\delta})+B_2\sinh(\xi\sqrt{-\delta})\right\}^2}{\left\{B_1\sinh(\xi\sqrt{-\delta})+B_2\cosh(\xi\sqrt{-\delta})+\frac{q}{\delta}\right\}^2}+\sqrt{-\delta\frac{(\delta^2S+q^2)}{\delta}}\frac{\left\{B_1\cosh(\xi\sqrt{-\delta})+B_2\sinh(\xi\sqrt{-\delta})\right\}}{\left\{B_1\sinh(\xi\sqrt{-\delta})+B_2\cosh(\xi\sqrt{-\delta})+\frac{q}{\delta}\right\}^2},$$

(2.1.12) where $\xi=x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha},k=a\delta+b\delta-3ac-3bd$. Therefore, the traveling wave solution in the space-time coordinate of equation (2.1.8) is:

$$\psi(x,y,t)=\delta-\delta\frac{\left\{B_1\cosh\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{-\delta}\right)+B_2\sinh\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{-\delta}\right)\right\}^2}{\left\{B_1\sinh\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{-\delta}\right)+B_2\cosh\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{-\delta}\right)+\frac{q}{\delta}\right\}^2}$$

$$+\sqrt{-\delta\frac{(\delta^2S+q^2)}{\delta}}\frac{\left\{B_1\cosh\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{-\delta}\right)+B_2\sinh\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{-\delta}\right)\right\}}{\left\{B_1\sinh\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{-\delta}\right)+B_2\cosh\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{-\delta}\right)+\frac{q}{\delta}\right\}^2}$$

(2.1.13) In a particular case, setting $B_2=0, B_1\neq 0$ and $q=0$ in (2.1.13), the solitary wave solution turns into

$$\psi(x,y,t)=\delta-\delta\coth^2\left(\sqrt{-\delta}\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\right)$$

$$\frac{+\delta\sqrt{S}}{A_1}\coth\left(\sqrt{-\delta}\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\right)$$

$$\times\operatorname{cosech}\left(\sqrt{-\delta}\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\right).$$

(2.1.14) Using the values of the parameters gathered in (2.1.11) into (2.1.9), we obtain the solution as

$$U(\xi)=\frac{2}{3}\delta-\delta\frac{\left\{B_1\cosh\left(\xi\sqrt{-\delta}\right)+B_2\sinh\left(\xi\sqrt{-\delta}\right)\right\}^2}{\left\{B_1\sinh\left(\xi\sqrt{-\delta}\right)+B_2\cosh\left(\xi\sqrt{-\delta}\right)+\frac{q}{\delta}\right\}^2}-\frac{q}{\left\{B_1\sinh\left(\xi\sqrt{-\delta}\right)+B_2\cosh\left(\xi\sqrt{-\delta}\right)+\frac{q}{\delta}\right\}^2}$$

$$+\sqrt{-\delta\frac{(\delta^2S+q^2)}{\delta}}\frac{\left\{B_1\cosh\left(\xi\sqrt{-\delta}\right)+B_2\sinh\left(\xi\sqrt{-\delta}\right)\right\}}{\left\{B_1\sinh\left(\xi\sqrt{-\delta}\right)+B_2\cosh\left(\xi\sqrt{-\delta}\right)+\frac{q}{\delta}\right\}^2},$$

(2.1.15) where $\xi=x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, k=-a\delta-b\delta-3ac-3bd$. Therefore, the closed-form wave solution in the space-time coordinates of equation (2.1.8) is

$$\psi(x,y,t)=\frac{2}{3}\delta-\delta\frac{\left\{B_1\cosh\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{-\delta}\right)+B_2\sinh\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{-\delta}\right)\right\}^2}{\left\{B_1\sinh\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{-\delta}\right)+B_2\cosh\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{-\delta}\right)+\frac{q}{\delta}\right\}^2}$$

$$\frac{-q}{\left\{B_1\sinh\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{-\delta}\right)+B_2\cosh\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{-\delta}\right)+\frac{q}{\delta}\right\}^2}$$

$$+\sqrt{-\delta\frac{(\delta^2 S+q^2)}{\delta}}\frac{\left\{B_1\cosh\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{-\delta}\right)+B_2\sinh\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{-\delta}\right)\right\}}{\left\{B_1\sinh\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{-\delta}\right)+B_2\cosh\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{-\delta}\right)+\frac{q}{\delta}\right\}^2}$$

(2.1.16) In particular, setting $B_1=0, B_2\neq 0$ and $q=0$ in (2.1.16), the solitary wave solution turns into

$$\psi(x,y,t)=\frac{2}{3}\delta-\delta\tanh^2\left(\sqrt{-\delta}\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\right)$$

$$\frac{+\delta\sqrt{S}}{B_2}\tanh\left(\sqrt{-\delta}\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\right)$$

$$\times\operatorname{sech}\left(\sqrt{-\delta}\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\right).$$

(2.1.17)

Case 2: If $\delta>0$. Introducing (2.1.9) into (2.1.8), using (1.3) and (1.7), the equation (1.8) becomes a polynomial in φ and ψ. Setting each coefficient of the polynomial to zero yields a group of algebraic equations and unraveling these algebraic equations provide the estimation of a_0, a_1, a_2, b_1, b_2 and k as follows:

$$a_0=\delta, a_1=0, a_2=1, b_1=-q, b_2=\sqrt{\frac{(\delta^2 S-q^2)}{\delta}},$$

$$k=a\delta+b\delta-3ac-3bd.$$

(2.1.18)

$$a_0=\frac{2}{3}\delta, a_1=0, a_2=1, b_1=-q, b_2=\sqrt{\frac{(\delta^2 S-q^2)}{\delta}},$$

$$k=-a\delta-b\delta-3ac-3bd.$$

(2.1.19) Using the values of the parameters gathered in equation (2.1.18) into (2.1.9), we find the solution as

$$U(\xi)=\delta+\frac{\delta\left\{B_1\cos(\xi\sqrt{\delta})-B_2\sin(\xi\sqrt{\delta})\right\}^2}{\left\{B_1\sin(\xi\sqrt{\delta})+B_2\cos(\xi\sqrt{\delta})+\frac{q}{\delta}\right\}^2}-\frac{q}{\left\{B_1\sin(\xi\sqrt{\delta})+B_2\cos(\xi\sqrt{\delta})+\frac{q}{\delta}\right\}^2}$$

$$\frac{+\sqrt{(\delta^2 S-q^2)}\left\{B_1\cos(\xi\sqrt{\delta})-B_2\sin(\xi\sqrt{\delta})\right\}}{\left\{B_1\sin(\xi\sqrt{\delta})+B_2\cos(\xi\sqrt{\delta})+\frac{q}{\delta}\right\}^2},$$

(2.1.20) where $\xi=x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, k=a\delta+b\delta-3ac-3bd$. Then the traveling wave solution in the space-time coordinate for equation (2.1.8) is

$$\psi(x,y,t)=\delta+\delta\frac{\left\{B_1\cos\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{\delta}\right)-B_2\sin\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{\delta}\right)\right\}^2}{\left\{B_1\sin\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{\delta}\right)+B_2\cos\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{\delta}\right)+\frac{q}{\delta}\right\}^2}$$

$$\frac{-q}{\left\{B_1\sin\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{\delta}\right)+B_2\cos\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{\delta}\right)+\frac{q}{\delta}\right\}^2}$$

$$\frac{+\sqrt{(\delta^2 S-q^2)}\left\{B_1\cos\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{\delta}\right)-B_2\sin\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{\delta}\right)\right\}}{\left\{B_1\sin\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{\delta}\right)+B_2\cos\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{\delta}\right)+\frac{q}{\delta}\right\}^2}$$

(2.1.21) In particular case, by setting $B_2=0, B_1\neq 0$ and $q=0$ in (2.1.21), the solitary wave solution

$$\psi(x,y,t)=\delta+\delta\left\{\cot^2\left(\sqrt{\delta}\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\right)\right\}$$

$$\frac{+\delta\sqrt{S}}{B_1}\cot\left(\sqrt{\delta}\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\right)$$

$$\times\operatorname{cosec}\left(\sqrt{\delta}\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\right).$$

(2.1.22) Again, using the values of the parameters gathered in equation (2.1.19) into (2.1.9), we find the solution as

$$U(\xi)=\frac{2}{3}\delta+\delta\frac{\left\{B_1\cos(\xi\sqrt{\delta})-B_2\sin(\xi\sqrt{\delta})\right\}^2}{\left\{B_1\sin(\xi\sqrt{\delta})+B_2\cos(\xi\sqrt{\delta})+\frac{q}{\delta}\right\}^2}-\frac{q}{\left\{B_1\sin(\xi\sqrt{\delta})+B_2\cos(\xi\sqrt{\delta})+\frac{q}{\delta}\right\}^2}$$

$$\frac{+\sqrt{(\delta^2 S-q^2)}\left\{B_1\cos(\xi\sqrt{\delta})-B_2\sin(\xi\sqrt{\delta})\right\}}{\left\{B_1\sin(\xi\sqrt{\delta})+B_2\cos(\xi\sqrt{\delta})+\frac{q}{\delta}\right\}^2},$$

(2.1.23) where $\xi = x + y - \frac{k}{a}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}, k = -a\delta - b\delta - 3ac - 3bd.$ Thus, the traveling wave solution in the space-time coordinate of equation (2.1.8) is

$$\psi(x,y,t) = \frac{2}{3}\delta + \delta\frac{\left\{B_1\cos\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{\delta}\right) - B_2\sin\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{\delta}\right)\right\}^2}{\left\{B_1\sin\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{\delta}\right) + B_2\cos\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{\delta}\right) + \frac{q}{\delta}\right\}^2}$$

$$\frac{-q}{\left\{B_1\sin\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{\delta}\right) + B_2\cos\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{\delta}\right) + \frac{q}{\delta}\right\}^2}$$

$$+\sqrt{(\delta^2 S - q^2)}\frac{\left\{B_1\cos\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{\delta}\right) - B_2\sin\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{\delta}\right)\right\}}{\left\{B_1\sin\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{\delta}\right) + B_2\cos\left(\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\sqrt{\delta}\right) + \frac{q}{\delta}\right\}^2}$$

(2.1.24) Moreover, by setting $B_1 = 0, B_2 \neq 0$ and $q = 0$, in (2.1.24), the solitary wave solution becomes

$$\psi(x,y,t) = \frac{2}{3}\delta + \delta\left\{\tan^2\left(\sqrt{\delta}\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\right)\right\}$$

$$\frac{+\delta\sqrt{S}}{B_2}\tan\left(\sqrt{\delta}\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\right)$$

$$\times sec\left(\sqrt{\delta}\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\right).$$

(2.1.25)

Case 3: If $\delta = 0$. Inserting (2.1.9) into (2.1.8), expending (1.3) and (1.9), the left hand side of (2.1.8) is induced to a polynomial φ and ψ. Setting the coefficients of the polynomial to zero gives a class of algebraic solutions and explaining these equations, it is attained two set values given in following:

$$a_0 = a_0, a_1 = 0, a_2 = a_2, b_1 = b_1, b_2 = 0,$$

$$k = 3b(-d+c), a = -b, c = c,$$

(2.1.26) and

$$a_0 = a_0, a_1 = a_1, a_2 = a_2, b_1 = b_1, b_2 = b_2,$$

$$k = 3bc - 3bd, a = -b, c = c.$$

(2.1.27) Using the values of the parameters gathered in equation (2.1.26) into (2.1.9), we find the solution as

$$U(\xi)=a_0+a_2\frac{\left(\frac{m\xi+B_1}{m\xi^2+B_1\xi+B_2}\right)^2}{2}+\frac{b_1}{\frac{m}{2}\xi^2+B_1\xi+B_2},$$

(2.1.28) where $\xi=x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, k=3b(-d+c)$. In the space-time coordinate, the wave solution for (2.1.8) is attained as

$$\psi(x,y,t)=a_0+a_2\frac{\left(\frac{q(x+y-\frac{k}{a}(t+\frac{1}{\Gamma(\alpha)})^{\alpha})+B_1}{\frac{q}{2}(x+y-\frac{k}{a}(t+\frac{1}{\Gamma(\alpha)})^{\alpha})^2+B_1(x+y-\frac{k}{a}(t+\frac{1}{\Gamma(\alpha)})^{\alpha})+B_2}\right)^2}{2}$$

$$+\frac{b_1}{\frac{q}{2}(x+y-\frac{k}{a}(t+\frac{1}{\Gamma(\alpha)})^{\alpha})^2+B_1(x+y-\frac{k}{a}(t+\frac{1}{\Gamma(\alpha)})^{\alpha})+B_2}$$

(2.1.29) Specifically, by setting $B_2=0, B_1\neq 0$ and $q=0$ in (2.1.29), the wave solution turns out to be

$$\psi(x,y,t)=a_0+\frac{a_2}{\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)^2}+\frac{b_1}{B_1\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)}$$

(2.1.30) Similarly, for the values of the parameters in (2.1.27) into (2.1.9), the wave solution we obtain is

$$U(\xi)=a_0+a_1\left(\frac{m\xi+B_1}{\frac{m}{2}\xi^2+B_1\xi+B_2}\right)+a_2\left(\frac{m\xi+B_1}{\frac{m}{2}\xi^2+B_1\xi+B_2}\right)^2$$

$$+\frac{b_1}{\frac{m}{2}\xi^2+B_1\xi+B_2}+b_2\frac{(m\xi+B_1)}{\left(\frac{m}{2}\xi^2+B_1\xi+B_2\right)^2}$$

(2.1.31) where $\xi=x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, k=3b(-d+c)$. Then the traveling wave solution in the space-time coordinate for equation (2.1.8) is

$$\psi(x,y,t)=a_0+a_1\left(\frac{m\left(x+y-\frac{k}{a}(t+\frac{1}{\Gamma(\alpha)})^{\alpha}\right)+B_1}{\frac{m}{2}\left(x+y-\frac{k}{a}(t+\frac{1}{\Gamma(\alpha)})^{\alpha}\right)^2+B_1\left(x+y-\frac{k}{a}(t+\frac{1}{\Gamma(\alpha)})^{\alpha}\right)+B_2}\right)$$

$$+a_2\left(\frac{m\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)+B_1}{\frac{m}{2}\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)^2+B_1\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)+B_2}\right)^2$$

$$\frac{+b_1}{\frac{m}{2}\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)^2+B_1\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)+B_2}$$

$$+b_2\frac{\left(m\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)+B_1\right)}{\left(\frac{m}{2}\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)^2+B_1\left(x+y-\frac{k}{a}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)+B_2\right)^2}$$

(2.1.32) Moreover, by setting $B_1=0, B_2\neq 0$ and $q=0$, in (2.1.32), the solitary wave solution becomes

$$\psi(x,y,t)=a_0+\frac{b_1}{B_2}$$

(2.1.33)

2.2. The space-time fractional nonlinear Schrödinger Kerr law equation

The fractional space-time perturbed nonlinear Schrödinger-Kerr law equation (Zhang et al., 2020; Inc et al., 2018; Ilyas et al., 2021; Wu and He, 2020; Liu et al., 2010; Zhang et al., 2014; Moosaeia et al., 2011) is given by

$$i\,D_t^{\alpha}\psi+a\,D_{xt}^{2\alpha}\psi+b\,D_x^{2\alpha}\psi+cF(\vee\psi¿^2)\psi=0,$$

(2.2.1) where *a* represents the coefficient of spatio-temporal dispersion, *b* signifies the coefficient for group velocity dispersion, and *c* denotes the coefficient for the nonlinear term. Consider the complex wave transformation

$$\psi(x,t)=U(\xi)e^{i\eta},$$

(2.2.2) where $\xi=\frac{1}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}-\frac{\omega}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \eta=\frac{k}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{w}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}$. Through the utilization of the complex wave transformation described in equation (2.2.2), the perturbed nonlinear Schrödinger equation is transformed into an equation, with its real and imaginary parts given as follows:

$$k^2(b-av)U''-(awk-w+bk^2)U+cF(\vee U¿^2)U=0.$$

(2.2.3)

$$v=\frac{aw-2bk}{1-ak}$$

(2.2.4) This nonlinearity is related to $F(U)=U$, from equation (2.2.3), we get

$$k^2(b-av)U''-(awk-w+bk^2)U+cU^3=0.$$

(2.2.5) Balancing between U'' and U^3, appearing in Eq. (2.2.5), we get $N=1$. Therefore, for equation (2.2.5), the trial solution can be written as

$$U(\xi)=a_0+a_1\varphi(\xi)+b_1\psi(\xi).$$

(2.2.6) By following the parallel procedures outlined in the methodology section, numerous soliton solutions can be established depending on a range of values of δ which are discussed below in detail:

Case 1: If ($\delta<0$), in this case, two sets of solutions are acquired, as presented below: Set 1: $a_0=0, a_1=\frac{\sqrt{-bk^2+k^2v}}{\sqrt{2c}}, b_1=\frac{k\sqrt{b-v}\sqrt{q^2+\delta^2S}}{\sqrt{2c}\delta}$,

$$w=\frac{1}{2}(2awk+2bk^2-bk^2\delta+k^2\delta v).$$

When the estimated values of parameters from Set 1 are introduced into solution (2.2.6), the solution $U(\xi)$ is determined as follows:

$$U(\xi)=\frac{\sqrt{-bk^2+k^2v}\sqrt{-\delta}\left\{B_1\cosh(\sqrt{-\delta}\xi)+B_2\sinh(\sqrt{-\delta}\xi)\right\}}{\sqrt{2c}\left\{B_1\sinh(\sqrt{-\delta}\xi)+B_2\cosh(\sqrt{-\delta}\xi)+\frac{q}{\delta}\right\}}$$

$$\frac{+k\sqrt{b-v}\sqrt{q^2+\delta^2S}}{\sqrt{2c}\delta\left\{B_1\sinh(\sqrt{-\delta}\xi)+B_2\cosh(\sqrt{-\delta}\xi)+\frac{q}{\delta}\right\}}$$

(2.2.7) along with $\xi=\frac{1}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}-\frac{\omega}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \eta=\frac{k}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{w}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}$. By setting (2.2.7) into (2.2.2), the analytical solution obtained for equation (2.2.1) is as follows:

$$\psi(x,t)=e^{i\eta}\left\{\frac{\sqrt{-bk^2+k^2v}\sqrt{-\delta}\left\{B_1\cosh(\sqrt{-\delta}\xi)+B_2\sinh(\sqrt{-\delta}\xi)\right\}}{\sqrt{2c}\left\{B_1\sinh(\sqrt{-\delta}\xi)+B_2\cosh(\sqrt{-\delta}\xi)+\frac{q}{\delta}\right\}}\right.$$

$$\left.+\frac{k\sqrt{b-v}\sqrt{q^2+\delta^2S}}{\sqrt{2c}\delta\left\{B_1\sinh(\sqrt{-\delta}\xi)+B_2\cosh(\sqrt{-\delta}\xi)+\frac{q}{\delta}\right\}}\right\}$$

(2.2.8) For $B_1=0, B_2\neq0$, and $q=0$, solution (2.2.8) can be modified into a widely acknowledged solution as:

$$\psi(x,t)=e^{i\eta}\left\{\sqrt{-\delta}\left(\frac{\sqrt{-bk^2+k^2v}}{\sqrt{2c}}\right)\tanh(\sqrt{-\delta}\xi)+\frac{k\sqrt{b-v}\sqrt{S}}{\sqrt{2c}B_2}\operatorname{sech}(\sqrt{-\delta}\xi)\right\}.$$

(2.2.9) On the other hand, for $B_1\neq0, B_2=0$, and $q=0$, the attained result turns into

$$\psi(x,t)=e^{i\eta}\left\{\sqrt{-\delta}\left(\frac{\sqrt{-bk^2+k^2v}}{\sqrt{2c}}\right)\coth\left(\sqrt{-\delta}\,\xi\right)+\frac{k\sqrt{b-v}\sqrt{S}}{\sqrt{2c}\,\delta B_1}\operatorname{cosech}\left(\sqrt{-\delta}\,\xi\right)\right\}.$$

(2.2.10) Set 2: $a_0=0, a_1=\frac{-\sqrt{-bk^2+k^2v}}{\sqrt{2c}}, b_1=\frac{-k\sqrt{b-v}\sqrt{q^2+\delta^2 S}}{\sqrt{2c}\,\delta},$

$$w=\frac{1}{2}\left(2awk+2bk^2-bk^2\delta+k^2\delta v\right).$$

The value of $U(\xi)$ is ascertained by introducing the estimated parameter values from Set 2 into equation (2.2.6).

$$U(\xi)=\frac{-k\sqrt{b-v}\sqrt{\delta}\left\{B_1\cosh\left(\sqrt{-\delta}\,\xi\right)+B_2\sinh\left(\sqrt{-\delta}\,\xi\right)\right\}}{\sqrt{2c}\left\{B_1\sinh\left(\sqrt{-\delta}\,\xi\right)+B_2\cosh\left(\sqrt{-\delta}\,\xi\right)+\frac{q}{\delta}\right\}}$$

$$\frac{+k\sqrt{b-v}\sqrt{q^2+\delta^2 S}}{\sqrt{2c}\,\delta\left\{B_1\sinh\left(\sqrt{-\delta}\,\xi\right)+B_2\cosh\left(\sqrt{-\delta}\,\xi\right)+\frac{q}{\delta}\right\}}$$

(2.2.11) wherein $\xi=\frac{1}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}-\frac{\omega}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \eta=\frac{k}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{w}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}$. By setting (2.2.7) into (2.2.2), the analytical solution for equation (2.2.1) is obtained as follows:

$$\psi(x,t)=e^{i\eta}\left\{\frac{-\sqrt{-bk^2+k^2v}\sqrt{\delta}\left\{B_1\cosh\left(\sqrt{-\delta}\,\xi\right)+B_2\sinh\left(\sqrt{-\delta}\,\xi\right)\right\}}{\sqrt{2c}\left\{B_1\sinh\left(\sqrt{-\delta}\,\xi\right)+B_2\cosh\left(\sqrt{-\delta}\,\xi\right)+\frac{q}{\delta}\right\}}\right.$$

$$\left.+\frac{k\sqrt{b-v}\sqrt{q^2+\delta^2 S}}{\sqrt{2c}\,\delta\left\{B_1\sinh\left(\sqrt{-\delta}\,\xi\right)+B_2\cosh\left(\sqrt{-\delta}\,\xi\right)+\frac{q}{\delta}\right\}}\right\}$$

(2.2.12) For $B_1=0, B_2\neq 0$, and $q=0$, solution (2.2.12) can be modified into a widely acknowledged solution as:

$$\psi(x,t)=e^{i\eta}\left\{\frac{k\sqrt{\delta(b-v)}}{\sqrt{2c}}\tanh\left(\sqrt{-\delta}\,\xi\right)+\frac{k\sqrt{S}\delta(b-v)}{\sqrt{2c}\,B_2}\operatorname{sech}\left(\sqrt{-\delta}\,\xi\right)\right\}.$$

(2.2.13) Similarly, for $B_1\neq 0, B_2=0$ and $q=0$, solution (2.2.12) can be obtained as:

$$\psi(x,t)=e^{i\eta}\left\{\frac{k\sqrt{\delta(b-v)}}{\sqrt{2c}}\coth\left(\sqrt{-\delta}\,\xi\right)+\frac{k\sqrt{S}\delta(b-v)}{\sqrt{2c}\,B_1}\operatorname{cosech}\left(\sqrt{-\delta}\,\xi\right)\right\}.$$

(2.2.14)

Case 2: If $\delta>0$, in this situation, two sets of solutions are obtained, and they are described below:

Set 1: $a_0=0, a_1=\frac{\sqrt{-bk^2+k^2v}}{\sqrt{2c}}, b_1=\frac{k\sqrt{q^2-S\delta^2}\sqrt{b-v}}{\sqrt{2c}\,\delta},$

$$w=\frac{1}{2}\left(2awk+k^2\left(2b-b\delta+\delta v\right)\right).$$

By substituting the specified parameter values into (2.2.6) and then using (2.2.2), we obtain the solution involving the trigonometric functions.

$$\psi(x,t)=e^{i\eta}\left\{\frac{-\sqrt{-bk^2+k^2v}\sqrt{\delta}\left[B_1\cos\left(\sqrt{\delta}\xi\right)-B_2\sin\left(\sqrt{\delta}\xi\right)\right]}{\sqrt{2c}\left\{B_1\sin\left(\sqrt{\delta}\xi\right)+B_2\cos\left(\sqrt{\delta}\xi\right)+\frac{q}{\delta}\right\}}\right.$$

$$\left.+\frac{k\sqrt{q^2-S\delta^2}\sqrt{b-v}}{\sqrt{2c}\,\delta\left\{B_1\sin\left(\sqrt{\delta}\xi\right)+B_2\cos\left(\sqrt{\delta}\xi\right)+\frac{q}{\delta}\right\}}\right\}.$$

(2.2.15) wherein $\xi=\frac{1}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}-\frac{\omega}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha},\eta=\frac{k}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{w}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}$. Putting $B_1=0, B_2\neq 0$ and $q=0$ in solution (2.2.15), we obtain

$$\psi(x,t)=e^{i\eta}\left\{\sqrt{\frac{-bk^2+k^2v}{2c}}\tan\left(\sqrt{\delta}\xi\right)-\frac{ik\delta\sqrt{b-v}\sqrt{S}}{\sqrt{2c}\,\delta B_2}\operatorname{cosec}\left(\sqrt{\delta}\xi\right)\right\}.$$

(2.2.16) Similarly, for $B_1\neq 0, B_2=0$ and $q=0$, we attain

$$\psi(x,t)=e^{i\eta}\left\{-\sqrt{\delta}\frac{\sqrt{-bk^2+k^2v}}{\sqrt{2c}}\cot\left(\sqrt{\delta}\xi\right)-\frac{ik\delta\sqrt{b-v}\sqrt{S}}{\sqrt{2c}\,\delta B_1}\operatorname{cosec}\left(\sqrt{\delta}\xi\right)\right\}.$$

(2.2.17) Set 2: $a_0=0, a_1=\frac{-\sqrt{-bk^2+k^2v}}{\sqrt{2c}}, b_1=\frac{-k\sqrt{q^2-S\delta^2}\sqrt{b-v}}{\sqrt{2c}\,\delta}$,

$$w=\frac{1}{2}\left(2awk+k^2\left(2b-b\delta+\delta v\right)\right).$$

Similarly, for parameter values of given Set 2, putting in equation (2.2.6), we get

$$U(\xi)=\frac{k\sqrt{\delta(v-b)}\left[B_1\cos\left(\sqrt{\delta}\xi\right)-B_2\sin\left(\sqrt{\delta}\xi\right)\right]}{\sqrt{2c}\left\{B_1\sin\left(\sqrt{\delta}\xi\right)+B_2\cos\left(\sqrt{\delta}\xi\right)+\frac{q}{\delta}\right\}}-\frac{k\sqrt{q^2-S\delta^2}\sqrt{b-v}}{\sqrt{2c}\,\delta\left\{B_1\sin\left(\sqrt{\delta}\xi\right)+B_2\cos\left(\sqrt{\delta}\xi\right)+\frac{q}{\delta}\right\}}$$

(2.2.18) Therefore, the soliton solution for equation (2.2.1), derived from (2.2.2), can be formulated as follows:

$$\psi(x,t)=e^{i\eta}\left\{\frac{k\sqrt{\delta(v-b)}\left[B_1\cos\left(\sqrt{\delta}\xi\right)-B_2\sin\left(\sqrt{\delta}\xi\right)\right]}{\sqrt{2c}\left\{B_1\sin\left(\sqrt{\delta}\xi\right)+B_2\cos\left(\sqrt{\delta}\xi\right)+\frac{q}{\delta}\right\}}\right.$$

$$\left.-\frac{k\sqrt{q^2-S\delta^2}\sqrt{b-v}}{\sqrt{2c}\,\delta\left\{B_1\sin\left(\sqrt{\delta}\xi\right)+B_2\cos\left(\sqrt{\delta}\xi\right)+\frac{q}{\delta}\right\}}\right\}$$

(2.2.19) when $B_1\neq 0, B_2=0$ and $q=0$, solution (2.2.19) yields

$$\psi(x,t)=e^{i\eta}\left\{\frac{k\sqrt{\delta(v-b)}}{\sqrt{2c}}\cot(\sqrt{\delta}\xi)-\frac{k\sqrt{S}\delta(v-b)}{B_1\sqrt{2c}}\operatorname{cosec}(\sqrt{\delta}\xi)\right\}.$$

(2.2.20) If $B_1=0, B_2\neq 0$ and $q=0$, solution (2.2.19) obtained

$$\psi(x,t)=e^{i\eta}\left\{\frac{-k\sqrt{\delta(v-b)}}{\sqrt{2c}}\tan(\sqrt{\delta}\xi)-\frac{k\sqrt{S}\delta(v-b)}{B_2\sqrt{2c}}\sec(\sqrt{\delta}\xi)\right\}.$$

(2.2.21)

Case 3: If $\delta=0$, repeating the former steps and unraveling the system of equations, we derive the ensuing values for the arbitrary constants:

$$a_0=a_0, a_1=a_1, b_1=b_1, \omega=awk+bk^2-ca_0^2.$$

By inserting these constraint values into solution (2.2.6), we obtain the resulting solution:

$$U(\xi)=a_0+\frac{2a_1(B_1+\xi)}{\xi^2+2B_1\xi+2B_2}+\frac{2b_1}{\xi^2+2B_1\xi+2B_2},$$

(2.2.22) where $\xi=\frac{k}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}-\frac{\omega}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \eta=\frac{k}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{w}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}$. Using the solution (2.2.22), from (2.2.2), we get the soliton solution of the successive form:

$$\psi(x,t)=e^{i\eta}\left(a_0+\frac{2a_1(B_1+\xi)}{\xi^2+2B_1\xi+2B_2}\right).$$

(2.2.23) Moreover, setting $B_1=0$ but $B_2\neq 0$, solution (2.2.23) turns out to be

$$\psi(x,t)=e^{i\eta}\left(a_0+\frac{2a_1\xi}{\xi^2+2B_2}\right).$$

(2.2.24) Similarly, for $B_1\neq 0$ but $B_2=0$, solution (2.2.23) becomes

$$\psi(x,t)=e^{i\eta}\left(a_0+\frac{2a_1(B_1+\xi)}{\xi^2+2B_1\xi}\right).$$

(2.2.25) For the other choice of the parameters it can be developed other type of rational solutions, but for the simplicity they are not reported here.

2.3. The space-time fractional unstable nonlinear Schrödinger equation

The fractional space-time unstable nonlinear Schrödinger (UNS) equation (Lu et al., 2018; Wazwaz, 2008; Sarwar et al., 2023; Tebue et al., 2016; Abdelrahman et al., 2020) is given by

$$iD_t^{\alpha}\psi+D_x^{2\alpha}\psi+2\lambda(\vee\psi¿^2\psi)-2\gamma\psi=0.$$

(2.3.1) We use the following transformations

$$\psi(x,t)=U(\xi)e^{i\eta},$$

(2.3.2) where $\xi=\frac{1}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{v}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \eta=\frac{\beta}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{r}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}$. With reference to equation (2.3.1), the space-time unstable nonlinear Schrödinger equation (2.3.2), turns out to be

$$U^{''}(\xi)-(\beta^2+r+2\gamma)U(\xi)+2\lambda U^3(\xi)=0,$$

(2.3.3) with $v=-2\beta;\beta$ is a constant which will be found later, while γ, r and λ are arbitrary constants. The balancing process of Eq. (2.3.3), gives $N=1$ and Eq. (1.13) becomes

$$U(\xi)=a_0+a_1\varphi(\xi)+b_1\psi(\xi).$$

(2.3.4) In the methodology, it describes the diverse solutions of the auxiliary equation. The solutions across different spectra for different values of δ are detailed below:

Case 1. ($\delta<0$): Here, two distinct sets of solutions emerge. Set 1: $a_0=0, a_1=\frac{i}{2\sqrt{\lambda}}, b_1=\frac{\sqrt{q^2+\delta^2 S}}{2\sqrt{\delta\lambda}}, r=\frac{1}{2}(\delta-2\beta^2-4\gamma)$. Solution (2.3.4) yields the following result when the parameters estimated in Set 1 are inserted:

$$U(\xi)=\frac{i\sqrt{-\delta}}{2\sqrt{\lambda}}\frac{\left\{B_1\cosh(\sqrt{-\delta}\,\xi)+B_2\sinh(\sqrt{-\delta}\,\xi)\right\}}{\left\{B_1\sinh(\sqrt{-\delta}\,\xi)+B_2\cosh(\sqrt{-\delta}\,\xi)+\frac{q}{\delta}\right\}}$$

$$\frac{+\sqrt{(q^2+\delta^2 S)}}{2\sqrt{\delta\lambda}\left\{B_1\sinh(\sqrt{-\delta}\,\xi)+B_2\cosh(\sqrt{-\delta}\,\xi)+\frac{q}{\delta}\right\}}$$

(2.3.5) With wave variable $\xi=\frac{1}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{v}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \eta=\frac{\beta}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{r}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}$. Therefore, the soliton solution for the fractional space-time unstable nonlinear Schrödinger equation (2.3.1) can be expressed as:

$$\psi(x,t)=e^{i\eta}\left\{\frac{i\sqrt{-\delta}}{2\sqrt{\lambda}}\frac{\left\{B_1\cosh(\sqrt{-\delta}\,\xi)+B_2\sinh(\sqrt{-\delta}\,\xi)\right\}}{\left\{B_1\sinh(\sqrt{-\delta}\,\xi)+B_2\cosh(\sqrt{-\delta}\,\xi)+\frac{q}{\delta}\right\}}\right.$$

$$\left.+\frac{\sqrt{(q^2+\delta^2 S)}}{2\sqrt{\delta\lambda}\left\{B_1\sinh(\sqrt{-\delta}\,\xi)+B_2\cosh(\sqrt{-\delta}\,\xi)+\frac{q}{\delta}\right\}}\right\}$$

(2.3.6) When $B_1=0, B_2\neq 0$, and $q=0$, solution (2.3.6) turns out to be the following soliton structure:

$$\psi(x,t)=e^{i\eta}\left\{\frac{i\sqrt{\delta}}{2\sqrt{\lambda}}\tanh(\sqrt{-\delta}\,\xi)+\frac{\sqrt{\delta S}}{2\sqrt{\lambda}B_2}\operatorname{sech}(\sqrt{-\delta}\,\xi)\right\}.$$

(2.3.7) When $B_2=0, B_1\neq 0$, and $q=0$, solution (2.3.6) yields the following result:

$$\psi(x,t)=e^{i\eta}\left\{\frac{i\sqrt{\delta}}{2\sqrt{\lambda}}\coth(\sqrt{-\delta}\,\xi)+\frac{\sqrt{\delta S}}{2\sqrt{\lambda}B_1}\operatorname{cosech}(\sqrt{-\delta}\,\xi)\right\}.$$

(2.3.8) Set 2: $a_0=0, a_1=\frac{-i}{2\sqrt{\lambda}}, b_1=\frac{-\sqrt{q^2+\delta^2 S}}{2\sqrt{\delta\lambda}}, r=\frac{1}{2}(\delta-2\beta^2-4\gamma)$. Inserting the parameters estimated in Set 2 into solution (2.3.4), the resulting outcome is as follows:

$$U(\xi)=\left\{\frac{-i\sqrt{-\delta}}{2\sqrt{\lambda}}\frac{\left\{B_1\cosh(\sqrt{-\delta}\,\xi)+B_2\sinh(\sqrt{-\delta}\,\xi)\right\}}{\left\{B_1\sinh(\sqrt{-\delta}\,\xi)+B_2\cosh(\sqrt{-\delta}\,\xi)+\frac{q}{\delta}\right\}}\right.$$

$$\left.-\frac{\sqrt{(q^2+\delta^2 S)}}{2\sqrt{\delta\lambda}\left\{B_1\sinh(\sqrt{-\delta}\,\xi)+B_2\cosh(\sqrt{-\delta}\,\xi)+\frac{q}{\delta}\right\}}\right\}$$

(2.3.9) Therefore, the soliton solution for the fractional space-time unstable nonlinear Schrödinger equation (2.3.2) can be expressed as:

$$\psi(x,t)=e^{i\eta}\left\{\frac{-i\sqrt{-\delta}}{2\sqrt{\lambda}}\frac{\left\{B_1\cosh(\sqrt{-\delta}\,\xi)+B_2\sinh(\sqrt{-\delta}\,\xi)\right\}}{\left\{B_1\sinh(\sqrt{-\delta}\,\xi)+B_2\cosh(\sqrt{-\delta}\,\xi)+\frac{q}{\delta}\right\}}\right.$$

$$\left.-\frac{\sqrt{(q^2+\delta^2 S)}}{2\sqrt{\delta\lambda}\left\{B_1\sinh(\sqrt{-\delta}\,\xi)+B_2\cosh(\sqrt{-\delta}\,\xi)+\frac{q}{\delta}\right\}}\right\}$$

(2.3.10) If the values for $B_1=0, B_2\neq 0$, and $q=0$ are used the solution (2.3.10) will be converted into

$$\psi(x,t)=e^{i\eta}\left\{\frac{i\sqrt{\delta}}{2\sqrt{\lambda}}\tanh(\sqrt{-\delta}\,\xi)-\frac{\sqrt{\delta S}}{2\sqrt{\lambda}B_2}\operatorname{sech}(\sqrt{-\delta}\,\xi)\right\}.$$

(2.3.11) By utilizing $B_1\neq 0, B_2=0$, and $q=0$, the solution (2.3.10) will be converted into

$$\psi(x,t)=e^{i\eta}\left\{\frac{i\sqrt{\delta}}{2\sqrt{\lambda}}\coth(\sqrt{-\delta}\,\xi)-\frac{\sqrt{\delta S}}{2\sqrt{\lambda}B_1}\operatorname{cosech}(\sqrt{-\delta}\,\xi)\right\}.$$

(2.3.12)

Case 2: When $\delta>0$, we obtain two sets of solutions, outlined as follows: Set 1: $a_0=0, a_1=\frac{i}{2\sqrt{\lambda}}, b_1=\frac{\sqrt{q^2-\delta^2 S}}{2\sqrt{\delta\lambda}}, r=\frac{1}{2}(\delta-2\beta^2-4\gamma)$. Putting the above solution Set 1 in (2.3.4) gives the following result:

$$U(\xi)=\left\{\frac{i\sqrt{\delta}}{2\sqrt{\lambda}}\frac{\left\{B_1\cos(\sqrt{\delta}\,\xi)-B_2\sin(\sqrt{\delta}\,\xi)\right\}}{\left\{B_1\sin(\sqrt{\delta}\,\xi)+B_2\cos(\sqrt{\delta}\,\xi)+\frac{q}{\delta}\right\}}\right.$$

$$\left.+\frac{\sqrt{q^2-\delta^2 S}}{2\sqrt{\delta\lambda}\left\{B_1\sin(\sqrt{\delta}\,\xi)+B_2\cos(\sqrt{\delta}\,\xi)+\frac{q}{\delta}\right\}}\right\}$$

(2.3.13) together with $\xi=\frac{1}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{\nu}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \eta=\frac{\beta}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{r}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}$. By using the transformation (2.3.2), we have obtained the exact soliton solution of Eq. (2.3.1) as follows:

$$\psi(x,t)=e^{i\eta}\left\{\frac{i\sqrt{\delta}}{2\sqrt{\lambda}}\frac{\left\{B_1\cos(\sqrt{\delta}\xi)-B_2\sin(\sqrt{\delta}\xi)\right\}}{\left\{B_1\sin(\sqrt{\delta}\xi)+B_2\cos(\sqrt{\delta}\xi)+\frac{q}{\delta}\right\}}\right.$$

$$\left.+\frac{\sqrt{q^2-\delta^2 S}}{2\sqrt{\delta\lambda}\left\{B_1\sin(\sqrt{\delta}\xi)+B_2\cos(\sqrt{\delta}\xi)+\frac{q}{\delta}\right\}}\right\}$$

(2.3.14) Setting $q=0, B_1=0$, and $B_2\neq 0$, the solution (2.3.14), reduces to

$$\psi(x,t)=e^{i\eta}\left\{\frac{-i\sqrt{\delta}}{2\sqrt{\lambda}}\tan(\sqrt{\delta}\xi)+\frac{i\sqrt{\delta S}}{2\sqrt{\lambda}B_2}\sec(\sqrt{\delta}\xi)\right\}.$$

(2.3.15) When $B_1\neq 0, B_2=0$, and $q=0$, the solution (2.3.14) produces the following form:

$$\psi(x,t)=e^{i\eta}\left\{\frac{i\sqrt{\delta}}{2\sqrt{\lambda}}\cot(\sqrt{\delta}\xi)-\frac{i\sqrt{\delta S}}{2\sqrt{\lambda}B_1}\operatorname{cosec}(\sqrt{\delta}\xi)\right\}.$$

(2.3.16) Set 2: $a_0=0, a_1=\frac{-i}{2\sqrt{\lambda}}, b_1=\frac{-\sqrt{q^2-\delta^2 S}}{2\sqrt{\delta\lambda}}, r=\frac{1}{2}(\delta-2\beta^2-4\gamma)$. Similarly, for the values of the parameters accumulated in Set 2, putting in equation (2.3.4) we attained

$$U(\xi)=\left\{\frac{-i\sqrt{\delta}}{2\sqrt{\lambda}}\frac{\left\{B_1\cos(\sqrt{\delta}\xi)-B_2\sin(\sqrt{\delta}\xi)\right\}}{\left\{B_1\sin(\sqrt{\delta}\xi)+B_2\cos(\sqrt{\delta}\xi)+\frac{q}{\delta}\right\}}\right.$$

$$\left.-\frac{\sqrt{q^2-\delta^2 S}}{2\sqrt{\delta\lambda}\left\{B_1\sin(\sqrt{\delta}\xi)+B_2\cos(\sqrt{\delta}\xi)+\frac{q}{\delta}\right\}}\right\}$$

(2.3.17) with $\xi=\frac{1}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{\nu}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \eta=\frac{\beta}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{r}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}$. We have obtained the exact soliton solution of Eq. (2.3.1), by using the transformation (2.3.2) as follows:

$$\psi(x,t)=e^{i\eta}\left\{\frac{-i\sqrt{\delta}}{2\sqrt{\lambda}}\frac{\left\{B_1\cos(\sqrt{\delta}\xi)-B_2\sin(\sqrt{\delta}\xi)\right\}}{\left\{B_1\sin(\sqrt{\delta}\xi)+B_2\cos(\sqrt{\delta}\xi)+\frac{q}{\delta}\right\}}\right.$$

$$\left.-\frac{\sqrt{q^2-\delta^2 S}}{2\sqrt{\delta\lambda}\left\{B_1\sin(\sqrt{\delta}\xi)+B_2\cos(\sqrt{\delta}\xi)+\frac{q}{\delta}\right\}}\right\}$$

(2.3.18) Putting $q=0, B_1=0$, and $B_2\neq 0$, the solution (2.3.18), reduces to

$$\psi(x,t)=e^{i\eta}\left\{\frac{i\sqrt{\delta}}{2\sqrt{\lambda}}\tan(\sqrt{\delta})-\frac{i\sqrt{\delta S}}{2\sqrt{\lambda}B_2}\sec(\sqrt{\delta}\xi)\right\}.$$

(2.3.19) When $B_1\neq 0, B_2=0$, and $q=0$, the solution (2.3.18) produces the following form:

$$\psi(x,t)=e^{i\eta}\left\{\frac{-i\sqrt{\delta}}{2\sqrt{\lambda}}\cot(\sqrt{\delta}\xi)-\frac{i\sqrt{\delta S}}{2\sqrt{\lambda}B_1}\operatorname{cosec}(\sqrt{\delta}\xi)\right\}.$$

(2.3.20)

Case 3: $\delta=0$: We have obtained one set of solutions in this case which is stated below:

$$a_0=a_0, a_1=a_1, b_1=b_1, r=\beta(4\gamma\beta-\lambda a_0^2).$$

By putting these values above in Eq. (2.3.4), we get the following result:

$$U(\xi)=a_0+\frac{2a_1(q\xi+B_1)}{q\xi^2+2B_1\xi+2B_2}+\frac{2b_1}{q\xi^2+2B_1\xi+2B_2},$$

(2.3.21) where $\xi=\frac{1}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{v}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \eta=\frac{\beta}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{r}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}$. By replacing transformation (2.3.2) in Eq. (2.3.21), we obtain the following result.

$$\psi(x,t)=e^{i\eta}\left\{a_0+\frac{2a_1(q\xi+B_1)}{q\xi^2+2B_1\xi+2B_2}+\frac{2b_1}{q\xi^2+2B_1\xi+2B_2}\right\}.$$

(2.3.22) Furthermore, if $B_1=0$ but $B_2\neq 0$, solution (2.3.22) becomes

$$\psi(x,t)=e^{i\eta}\left\{a_0+\frac{2a_1q\xi}{q\xi^2+2B_2}+\frac{2b_1}{q\xi^2+2B_2}\right\}.$$

(2.3.23) Likewise, when $B_1\neq 0$ but $B_2=0$, solution (2.3.22) yields

$$\psi(x,t)=e^{i\eta}\left\{a_0+\frac{2a_1(q\xi+B_1)}{q\xi^2+2B_1\xi}+\frac{2b_1}{q\xi^2+2B_1\xi}\right\}.$$

(2.3.24) Similarly, various rich structure soliton solutions can be derived for alternative values of the integral parameters. However, to ensure terseness, we have omitted any additional solutions beyond those already discussed.

2.4. The space-time fractional Schrödinger-Hirota equation

The space-time fractional perturbed nonlinear Schrödinger-Hirota equation (Ekici et al., 2017; Sharif, 2023; Biswas et al., 2012; Onder et al., 2024; Ray, 2020) is considered as

$$iD_t^{\alpha}\psi+aD_x^{2\alpha}\psi+bD_{xt}^{2\alpha}\psi+c\vee\psi¿^2\psi+i(\gamma D_{xxx}^{3\alpha}\psi+\theta\vee\psi¿^2D_x^{\alpha}\psi)=i\mu D_t^{3\alpha}\psi+i\lambda D_x^{3\alpha}(\vee\psi¿^2\psi)+i\nu D_x^{3\alpha}(\vee\psi¿^2\psi).$$

(2.4.1) Consider the complex transformation

$$\psi(x,t)=U(\xi)e^{i\eta},$$

(2.4.2) where $\xi=\frac{1}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}-\frac{k}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \eta=\frac{k}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{\omega}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}$. Here, v represents the soliton velocity, k is the constant soliton frequency, ω is soliton wave number. Substituting transformation (2.4.2) into (2.4.1) and equating real and imaginary parts, we acquire

$$(a-bv\omega+3\gamma k)U''-(\omega+b\omega k+\mu k^2+\gamma k^3+\mu k)U+(\lambda k-\theta k-c)U^3=0.$$

(2.4.3)

$$\gamma U'''-\{v+\mu+2\mu k+b\omega(\omega+vk)+3\gamma k^2\}U'+(\theta-3\lambda-2v)U^2U'=0.$$

(2.4.4) Since $\gamma=0$. (2.4.5) Therefore, Eq. (2.4.3) and (2.4.4), by virtue of Eq. (2.4.5), reduces to

$$(a-bv\omega)U''-(\omega+b\omega k+\mu k^2+\mu k)U+(\lambda k-\theta k-c)U^3=0.$$

(2.4.6)

$$\{v+\mu+2\mu k+b\omega(\omega+vk)\}U'-(\theta-3\lambda-2v)U^2U'=0.$$

(2.4.7) The remaining linearly independent functions, from Eq. (2.4.4), lead to the constraint

$$\theta=3\lambda+2v.$$

(2.4.8) and the speed of the soliton

$$v=\frac{\mu+2ak+b\omega^2}{1+b\omega k}$$

(2.4.9) The remaining Eq. (2.4.6) is supposed to be resolved. Thus by the application of homogeneous balancing principle in Eq. (2.4.6), the balanced parameter has been accomplished as $N=1$. Therefore, the trial solution of Eq. (2.4.6) is acknowledged in the following form:

$$U(\xi)=a_0+a_1\varphi(\xi)+b_1\psi(\xi).$$

(2.4.10) Executing the steps described comprehensively in the methodology section, abundant soliton solutions have been generated depending on a variety of values δ which are explained thoroughly as follows:

Case 1 ($\delta<0$): In this instance, two solution sets of the described equation are obtained for different possible conditions. Set 1: $a_0=a_0, a_1=a_1, b_1=b_1, \omega=\frac{c\mu(c\lambda-\theta)}{(\lambda-\theta)(bc+\lambda-\theta)}, k=\frac{c}{\lambda-\theta}$. By introducing the estimated values of parameters arranged in set 1 to solution (2.4.10), the values of $q(\xi)$ are as follows:

$$U(\xi)=a_0+a_1\frac{\sqrt{-\delta}\left\{B_1\cosh(\sqrt{-\delta}\,\xi)+B_2\sinh(\sqrt{-\delta}\,\xi)\right\}}{B_1\sinh(\sqrt{-\delta}\,\xi)+B_2\cosh(\sqrt{-\delta}\,\xi)+\frac{q}{\delta}}$$

$$+\frac{b_1}{B_1\sinh(\sqrt{-\delta}\,\xi)+B_2\cosh(\sqrt{-\delta}\,\xi)+\frac{q}{\delta}}$$

(2.4.11) along with $\xi=\frac{1}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}-\frac{v}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha},\eta=\frac{k}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{\omega}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}$. By setting Eq. (2.4.11) in (2.4.2), the required analytical solution of Eq. (2.4.1) is obtained as follows:

$$\psi(x,t)=e^{i\eta}\left\{a_0+a_1\frac{\sqrt{-\delta}\left\{B_1\cosh\left(\sqrt{-\delta}\,\xi\right)+B_2\sinh\left(\sqrt{-\delta}\,\xi\right)\right\}}{B_1\sinh\left(\sqrt{-\delta}\,\xi\right)+B_2\cosh\left(\sqrt{-\delta}\,\xi\right)+\frac{q}{\delta}}\right.$$

$$\left.+\frac{b_1}{B_1\sinh\left(\sqrt{-\delta}\,\xi\right)+B_2\cosh\left(\sqrt{-\delta}\,\xi\right)+\frac{q}{\delta}}\right\}$$

(2.4.12) Solution (2.4.12) can be transformed into a well-known solution for $B_1=0,B_2\neq 0$ and $q=0$ as

$$\psi(x,t)=e^{i\eta}\left\{a_0+a_1\sqrt{-\delta}\tanh\left(\sqrt{-\delta}\,\xi\right)+\frac{b_1}{B_2}\operatorname{sech}\left(\sqrt{-\delta}\,\xi\right)\right\}.$$

(2.4.13) On the other hand, the attained result for $B_1\neq 0,B_2=0$ and $q=0$ is

$$\psi(x,t)=e^{i\eta}\left\{a_0+a_1\sqrt{-\delta}\coth\left(\sqrt{-\delta}\,\xi\right)+\frac{b_1}{B_1}\operatorname{cosech}\left(\sqrt{-\delta}\,\xi\right)\right\}.$$

(2.4.14) Set 2: $a_0=0,a_1=a_1,b_1=b_1,\omega=\frac{c\mu(c\lambda-\theta)}{(\lambda-\theta)(bc+\lambda-\theta)},k=\frac{c}{\lambda-\theta}$. Setting the estimation of the parameters arranged in set 2, equation (2.4.10) provides the result:

$$U(\xi)=\frac{a_1\sqrt{-\delta}\left\{B_1\cosh\left(\sqrt{-\delta}\,\xi\right)+B_2\sinh\left(\sqrt{-\delta}\,\xi\right)\right\}}{B_1\sinh\left(\sqrt{-\delta}\,\xi\right)+B_2\cosh\left(\sqrt{-\delta}\,\xi\right)+\frac{q}{\delta}}$$

$$\frac{+b_1}{B_1\sinh\left(\sqrt{-\delta}\,\xi\right)+B_2\cosh\left(\sqrt{-\delta}\,\xi\right)+\frac{q}{\delta}}$$

(2.4.15) Hence, the desired solution of Eq. (2.4.15) is derived from transformation of Eq. (2.4.2).

$$\psi(x,t)=e^{i\eta}\left\{\frac{a_1\sqrt{-\delta}\left\{B_1\cosh\left(\sqrt{-\delta}\,\xi\right)+B_2\sinh\left(\sqrt{-\delta}\,\xi\right)\right\}}{B_1\sinh\left(\sqrt{-\delta}\,\xi\right)+B_2\cosh\left(\sqrt{-\delta}\,\xi\right)+\frac{q}{\delta}}\right.$$

$$\left.+\frac{b_1}{B_1\sinh\left(\sqrt{-\delta}\,\xi\right)+B_2\cosh\left(\sqrt{-\delta}\,\xi\right)+\frac{q}{\delta}}\right\}$$

(2.4.16) If, we assign the values $B_1=0,B_2\neq 0$ and $q=0$, the solution (2.4.16) will be transformed into

$$\psi(x,t)=e^{i\eta}\left\{a_1\sqrt{-\delta}\tanh\left(\sqrt{-\delta}\,\xi\right)+\frac{b_1}{B_2}\operatorname{sech}\left(\sqrt{-\delta}\,\xi\right)\right\}.$$

(2.4.17) Similarly, for $B_1 \neq 0, B_2 = 0$ and $q = 0$, the solution (2.4.16) takes the form

$$\psi(x,t) = e^{i\eta}\left\{a_1\sqrt{-\delta}\coth\left(\sqrt{-\delta}\,\xi\right) + \frac{b_1}{B_1}\operatorname{cosech}\left(\sqrt{-\delta}\,\xi\right)\right\}.$$

(2.4.18)

Case 2: If ($\delta > 0$), in this case, two sets of solutions have been identified and they are as follows: Set 1: $a_0 = a_0, a_1 = a_1, b_1 = 0, \omega = \frac{c\mu(c\lambda-\theta)}{(\lambda-\theta)(bc+\lambda-\theta)}, k = \frac{c}{\lambda-\theta}$. By setting parametric values, stated above, in Eq. (2.4.10) and Eq. (2.4.2), the subsequent solution involving trigonometric functions are attained.

$$\psi(x,t) = e^{i\eta}\left\{a_0 + a_1\frac{i\sqrt{\delta}\left\{B_1\cos\left(\sqrt{\delta}\,\xi\right) - B_2\sin\left(\sqrt{\delta}\,\xi\right)\right\}}{B_1\sin\left(\sqrt{\delta}\,\xi\right) + B_2\cos\left(\sqrt{\delta}\,\xi\right) + \frac{q}{\delta}}\right\}$$

(2.4.19) wherein $\xi = \frac{1}{\alpha}\left(x + \frac{1}{\Gamma(\alpha)}\right)^{\alpha} - \frac{v}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \eta = \frac{k}{\alpha}\left(x + \frac{1}{\Gamma(\alpha)}\right)^{\alpha} + \frac{\omega}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}$. Putting $B_1 = 0, B_2 \neq 0$ and $q = 0$ in solution (2.4.19), we obtain

$$\psi(x,t) = e^{i\eta}\left\{a_0 - a_1\sqrt{\delta}\tan\left(\sqrt{\delta}\,\xi\right)\right\}.$$

(2.4.20) Similarly, for $B_1 \neq 0, B_2 = 0$ and $q = 0$, we attain

$$\psi(x,t) = e^{i\eta}\left\{a_0 + a_1\sqrt{\delta}\cot\left(\sqrt{\delta}\,\xi\right)\right\}.$$

(2.4.21) Set 2: $a_0 = 0, a_1 = a_1, b_1 = 0, \omega = \frac{c\mu(c\lambda-\theta)}{(\lambda-\theta)(bc+\lambda-\theta)}, k = \frac{c}{\lambda-\theta}$. In a similar manner, for the values of the parameters in set 2, putting in Eq. (2.4.10). We obtained.

$$U(\xi) = a_1\frac{i\sqrt{\delta}\left\{B_1\cos\left(\sqrt{\delta}\,\xi\right) - B_2\sin\left(\sqrt{\delta}\,\xi\right)\right\}}{B_1\sin\left(\sqrt{\delta}\,\xi\right) + B_2\cos\left(\sqrt{\delta}\,\xi\right) + \frac{q}{\delta}}$$

(2.4.22) Hence, the desired solution of Eq. (2.4.22) is derived from transformation of Eq. (2.4.2).

$$\psi(x,t) = e^{i\eta}\left\{a_1\frac{i\sqrt{\delta}\left\{B_1\cos\left(\sqrt{\delta}\,\xi\right) - B_2\sin\left(\sqrt{\delta}\,\xi\right)\right\}}{B_1\sin\left(\sqrt{\delta}\,\xi\right) + B_2\cos\left(\sqrt{\delta}\,\xi\right) + \frac{q}{\delta}}\right\}$$

(2.4.23) If $B_1 = 0, B_2 \neq 0$ and $q = 0$, the required solution of Eq. (2.4.23) is

$$\psi(x,t) = e^{i\eta}\left\{a_1\sqrt{\delta}\cot\left(\sqrt{\delta}\,\xi\right)\right\}.$$

(2.4.24) Similarly, for $B_1 \neq 0, B_2 = 0$ and $q = 0$, we obtain

$$\psi(x,t) = e^{i\eta}\left\{-a_1\sqrt{\delta}\tan\left(\sqrt{\delta}\,\xi\right)\right\}.$$

(2.4.25)

Case 3: If ($\delta=0$), we obtain the following values of the arbitrary constants by reiterating the preceding measures and solving the system of equations:

$$a_0=a_0, a_1=a_1, b_1=0, \omega=\frac{c\mu(c\lambda-\theta)}{(\lambda-\theta)(bc+\lambda-\theta)}, k=\frac{c}{\lambda-\theta}.$$

Inserting these values of the constraints presented in Eq. (2.4.10), we attain the ensuring solution:

$$U(\xi)=a_0+\frac{2a_1(B_1+\xi)}{\xi^2+2B_1\xi+2B_2},$$

(2.4.26) with $\xi=\frac{1}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}-\frac{\nu}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \eta=\frac{k}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{\omega}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}$. From the Eq. (2.4.26) and (2.4.2), we get the soliton solution of the successive form:

$$\psi(x,t)=e^{i\eta}\left(a_0+\frac{2a_1(B_1+\xi)}{\xi^2+2B_1\xi+2B_2}\right).$$

(2.4.27) Moreover, setting $B_1=0$ but $B_2\neq0$, solution (2.4.27) turns out to be

$$\psi(x,t)=e^{i\eta}\left(a_0+\frac{2a_1\xi}{\xi^2+2B_2}\right).$$

(2.4.28) Similarly, for $B_1\neq0$ but $B_2=0$, solution (2.4.27) becomes

$$\psi(x,t)=e^{i\eta}\left(a_0+\frac{2a_1(B_1+\xi)}{\xi^2+2B_1\xi}\right).$$

(2.4.29)

2.5. The space-time fractional Biswas-Arshed equation

The time-space fractional Biswas-Arshed equation (Akbar et al., 2012; Bilal et al., 2022; Zayed and Shohib, 2019; Akbulut and Islam, 2022; Ganie et al., 2024;Yildirim, 2019) with beta fractional derivative is as follows:

$$i D_t^{\alpha}\psi+a_1 D_x^{2\alpha}\psi+a_2 D_t^{2\alpha}\psi+i\left(b_1 D_{xxt}^{3\alpha}\psi+b_2 D_{ttt}^{3\alpha}\psi\right)-i[\lambda D_x^{\alpha}(\vee\psi¿^2\psi)+\nu D_x^{\alpha}(\vee\psi¿^2)\psi+\theta\vee\psi¿^2 D_x^{\alpha}\psi]=0.$$

(2.5.1) Here, with $0<\alpha<1$ describing the order of fractional derivative, the wave potential is denoted as $\psi=\psi(x,t)$, where $\psi(x,t)$ is considered to be a complex-valued function of the spatial variable x and the temporal variable t. The derivatives $D_t^{\alpha}\psi$ and $D_x^{\alpha}\psi$ represent the beta fractional derivatives. In equation (2.5.1), the first term characterizes the propagation of optical pulses, with the coefficients a_1 and a_2 denoting group velocity dispersion and spatio-temporal dispersion respectively. The subsequent nonlinear terms, representing self-steepening and nonlinear dispersions, are controlled sequentially by the coefficients λ, ν, and θ. Additionally, the presence of third-order dispersion and third-order spatiotemporal dispersion is assured by the coefficients b_1 and b_2, respectively. We utilize this established model to explore novel characteristics of the evolution of soliton solutions on varying the fractional order values. We consider the fractional wave transformation

$$\psi(x,t)=U(\xi)e^{i\varphi},$$

(2.5.2) where $\xi=\frac{1}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}-\frac{\rho}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \varphi=\frac{-k}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{\omega}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}$. where ρ, k, ω are constants. By relating the transformation (2.5.2) to (2.5.1), cancelling $e^{i\varphi}$, and splitting the real and imaginary components, we obtain the following result:

$$(a_1-\rho b_1+3kc-\omega d-2k\rho d)U''-(\omega+ak^2-k\omega b-k^2\omega d+ck^3)U$$
$$-k(\lambda+\theta)U^3=0.$$

(2.5.3)

$$(c-\rho d)U''+(-3k^2c-2ka+\omega b+2k\omega d+k^2\rho d-\rho+k\rho b)U$$
$$\frac{-(2\theta+v+3\lambda)}{3}U^3=0.$$

(2.5.4) Balancing U'' and U^3 in Eq. (2.5.3), gives $3N=N+2$, implies $N=1$. Thus, equation (2.5.3) possesses a solution in the following form:

$$U(\xi)=a_0+a_1\varphi(\xi)+b_1\psi(\xi).$$

(2.5.5) wherein a_0, a_1, b_1 are under determined constants.

Case 1: For ($\delta<0$), by inserting equation (2.5.5) into equation (2.5.3) and following the steps outlined in the methodology, several solutions have been obtained for different values of δ. These solutions can be categorized into two sets, which are explained below: Set 1: $a_0=0, a_1=\sqrt{\frac{a+3kc-2k\rho d-\rho b-\omega d}{2(k\theta+k\lambda)}}, b_1=\frac{i\sqrt{q^2+\delta^2}S\sqrt{a+3kc-2k\rho d-\rho b-\omega d}}{\sqrt{2k\delta\theta+2k\delta\lambda}}$,

$$\omega=\frac{1}{2}(-2ak^2-2ck^3+2k\omega b+a\delta+3kc\delta-2k\rho d\delta-\delta\rho b+2k^2\omega d-\delta\omega d).$$

If we substitute the above-mentioned set of values into solution (2.5.5), we obtain the following result:

$$U(\xi)=\sqrt{\frac{a+3kc-2k\rho d-\rho b-\omega d}{2(k\theta+k\lambda)}}\frac{\sqrt{-\delta}\left\{B_1\cosh(\sqrt{-\delta}\xi)+B_2\sinh(\sqrt{-\delta}\xi)\right\}}{B_1\sinh(\sqrt{-\delta}\xi)+B_2\cosh(\sqrt{-\delta}\xi)+\frac{q}{\delta}}$$
$$+i\sqrt{q^2+\delta^2}S\sqrt{a+3kc-2k\rho d-\rho b-\omega d}$$
$$\sqrt{2k\delta\theta+2k\delta\lambda}\left\{B_1\sinh(\sqrt{-\delta}\xi)+B_2\cosh(\sqrt{-\delta}\xi)+\frac{q}{\delta}\right\}$$

(2.5.6) Therefore, the soliton solution for equation (2.5.1), derived from (2.5.2), can be formulated as follows:

$$\psi(x,t)=e^{i\eta}\left\{\sqrt{\frac{a+3kc-2k\rho d-\rho b-\omega d}{2(k\theta+k\lambda)}}\frac{\sqrt{-\delta}\left\{B_1\cosh(\sqrt{-\delta}\xi)+B_2\sinh(\sqrt{-\delta}\xi)\right\}}{B_1\sinh(\sqrt{-\delta}\xi)+B_2\cosh(\sqrt{-\delta}\xi)+\frac{q}{\delta}}\right.$$

$$+\frac{i\sqrt{q^2+\delta^2 S}\sqrt{a+3kc-2k\rho d-\rho b-\omega d}}{\sqrt{2k\delta\theta+2k\delta\lambda}\left\{B_1\sinh\left(\sqrt{-\delta}\,\xi\right)+B_2\cosh\left(\sqrt{-\delta}\,\xi\right)+\frac{q}{\delta}\right\}}\Bigg\}$$

(2.5.7) where $\xi=\frac{1}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}-\frac{\rho}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \eta=\frac{-k}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{\omega}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}$. If we take $B_1=0, B_2\neq 0$ and $q=0$ solution (2.5.7) will be altered to

$$\psi(x,t)=e^{i\eta}\Bigg\{\sqrt{-\delta}\left(\frac{a+3kc-2k\rho d-\rho b-\omega d}{2(k\theta+k\lambda)}\right)\tanh\left(\sqrt{-\delta}\,\xi\right)$$

$$+\frac{i\delta\sqrt{S}\sqrt{a+3kc-2k\rho d-\rho b-\omega d}}{B_2\sqrt{2k\delta\theta+2k\delta\lambda}}\operatorname{sech}\left(\sqrt{-\delta}\,\xi\right)\Bigg\}.$$

(2.5.8) Similarly, for $B_2=0, B_1\neq 0$ and $q=0$ the solution (2.5.7) takes the form

$$\psi(x,t)=e^{i\eta}\Bigg\{\sqrt{-\delta}\left(\frac{a+3kc-2k\rho d-\rho b-\omega d}{2(k\theta+k\lambda)}\right)\coth\left(\sqrt{-\delta}\,\xi\right)$$

$$+\frac{i\delta\sqrt{S}\sqrt{a+3kc-2k\rho d-\rho b-\omega d}}{B_1\sqrt{2k\delta\theta+2k\delta\lambda}}\operatorname{cosech}\left(\sqrt{-\delta}\,\xi\right)\Bigg\}.$$

(2.5.9) Set 2: $a_0=0, a_1=-\sqrt{\frac{a+3kc-2k\rho d-\rho b-\omega d}{2(k\theta+k\lambda)}}, b_1=\frac{-i\sqrt{q^2+\delta^2 S}\sqrt{a+3kc-2k\rho d-\rho b-\omega d}}{\sqrt{2k\delta\theta+2k\delta\lambda}}$,

$$\omega=\frac{1}{2}\left(-2ak^2-2ck^3+2k\omega b+a\delta+3kc\delta-2k\rho d\delta-\delta\rho b+2k^2\omega d-\delta\omega d\right).$$

Applying the values assembled in Set 2 mentioned above to solution (2.5.9), we obtain the resulting solution as follows:

$$U(\xi)=-\sqrt{\frac{a+3kc-2k\rho d-\rho b-\omega d}{2(k\theta+k\lambda)}}\,\frac{\sqrt{-\delta}\left\{B_1\cosh\left(\sqrt{-\delta}\,\xi\right)+B_2\sinh\left(\sqrt{-\delta}\,\xi\right)\right\}}{B_1\sinh\left(\sqrt{-\delta}\,\xi\right)+B_2\cosh\left(\sqrt{-\delta}\,\xi\right)+\frac{q}{\delta}}$$

$$\frac{+i\sqrt{q^2+\delta^2 S}\sqrt{a+3kc-2k\rho d-\rho b-\omega d}}{\sqrt{2k\delta\theta+2k\delta\lambda}\left\{B_1\sinh\left(\sqrt{-\delta}\,\xi\right)+B_2\cosh\left(\sqrt{-\delta}\,\xi\right)+\frac{q}{\delta}\right\}}$$

(2.5.10) Hence, transformation (2.5.2) enabled us to formulate the soliton solution of equation (2.5.1) in the subsequent form:

$$\psi(x,t)=e^{i\eta}\Bigg\{-\sqrt{\frac{a+3kc-2k\rho d-\rho b-\omega d}{2(k\theta+k\lambda)}}\,\frac{\sqrt{-\delta}\left\{B_1\cosh\left(\sqrt{-\delta}\,\xi\right)+B_2\sinh\left(\sqrt{-\delta}\,\xi\right)\right\}}{B_1\sinh\left(\sqrt{-\delta}\,\xi\right)+B_2\cosh\left(\sqrt{-\delta}\,\xi\right)+\frac{q}{\delta}}$$

$$+\frac{i\sqrt{q^2+\delta^2 S}\sqrt{a+3kc-2k\rho d-\rho b-\omega d}}{\sqrt{2k\delta\theta+2k\delta\lambda}\left\{B_1\sinh\left(\sqrt{-\delta}\,\xi\right)+B_2\cosh\left(\sqrt{-\delta}\,\xi\right)+\frac{q}{\delta}\right\}}\Bigg\}$$

(2.5.11) with $\xi=\frac{1}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}-\frac{\rho}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \eta=\frac{-k}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{\omega}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}$. If we set the value for $B_1=0, B_2\neq 0$ and $q=0$ the solution (2.5.11) will be transformed into

$$\psi(x,t)=e^{i\eta}\left\{\sqrt{-\delta}\left(\frac{a+3kc-2k\rho d-\rho b-\omega d}{2(k\theta+k\lambda)}\right)\tanh\left(\sqrt{-\delta}\,\xi\right)\right.$$

$$\left.+\frac{i\delta\sqrt{S}\sqrt{a+3kc-2k\rho d-\rho b-\omega d}}{B_2\sqrt{2k\delta\theta+2k\delta\lambda}}\operatorname{sech}\left(\sqrt{-\delta}\,\xi\right)\right\}.$$

(2.5.12) For $B_1\neq 0, B_2=0$ and $q=0$, an alternative solution can be obtained from (2.5.11) in the similar way:

$$\psi(x,t)=e^{i\eta}\left\{\sqrt{-\delta}\left(\frac{a+3kc-2k\rho d-\rho b-\omega d}{2(k\theta+k\lambda)}\right)\coth\left(\sqrt{-\delta}\,\xi\right)\right.$$

$$\left.+\frac{i\delta\sqrt{S}\sqrt{a+3kc-2k\rho d-\rho b-\omega d}}{B_1\sqrt{2k\delta\theta+2k\delta\lambda}}\operatorname{cosec}\left(\sqrt{-\delta}\,\xi\right)\right\}.$$

(2.5.13)

Case 2: When ($\delta>0$), the following two sets of potential solutions for the subjective parameters are obtained: Set 1: $a_0=0, a_1=\sqrt{\frac{a+3kc-2k\rho d-\rho b-\omega d}{2(k\theta+k\lambda)}}, b_1=\frac{i\sqrt{q^2-\delta^2 S}\sqrt{a+3kc-2k\rho d-\rho b-\omega d}}{\sqrt{2k\delta\theta+2k\delta\lambda}}$

$$\omega=\frac{1}{2}\left(-2ak^2-2ck^3+2k\omega b+a\delta+3kc\delta-2k\rho d\delta-\delta\rho b+2k^2\omega d-\delta\omega d\right).$$

When the parameter values in Set 1 are introduced, solution (2.5.5) provides the estimation:

$$U(\xi)=\sqrt{\frac{a+3kc-2k\rho d-\rho b-\omega d}{2(k\theta+k\lambda)}}\,\frac{i\sqrt{\delta}\left\{B_1\cos\left(\sqrt{\delta}\,\xi\right)-B_2\sin\left(\sqrt{\delta}\,\xi\right)\right\}}{B_1\sin\left(\sqrt{\delta}\,\xi\right)+B_2\cos\left(\sqrt{\delta}\,\xi\right)+\frac{q}{\delta}}$$

$$\frac{+i\sqrt{q^2-\delta^2 S}\sqrt{a+3kc-2k\rho d-\rho b-\omega d}}{\sqrt{2k\delta\theta+2k\delta\lambda}\left\{B_1\sin\left(\sqrt{\delta}\,\xi\right)+B_2\cos\left(\sqrt{\delta}\,\xi\right)+\frac{q}{\delta}\right\}}$$

(2.5.14) wherein $\xi=\frac{1}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}-\frac{\rho}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \eta=\frac{-k}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{\omega}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}$. Therefore, setting (2.5.14) into (2.5.2), we attain the subsequent solution of the Biswas-Arshed equation (2.5.1):

$$\psi(x,t)=e^{i\eta}\left\{\sqrt{\frac{a+3kc-2k\rho d-\rho b-\omega d}{2(k\theta+k\lambda)}}\,\frac{i\sqrt{\delta}\left\{B_1\cos\left(\sqrt{\delta}\,\xi\right)-B_2\sin\left(\sqrt{\delta}\,\xi\right)\right\}}{B_1\sin\left(\sqrt{\delta}\,\xi\right)+B_2\cos\left(\sqrt{\delta}\,\xi\right)+\frac{q}{\delta}}\right.$$

$$\left.+\frac{i\sqrt{q^2-\delta^2 S}\sqrt{a+3kc-2k\rho d-\rho b-\omega d}}{\sqrt{2k\delta\theta+2k\delta\lambda}\left\{B_1\sin\left(\sqrt{\delta}\,\xi\right)+B_2\cos\left(\sqrt{\delta}\,\xi\right)+\frac{q}{\delta}\right\}}\right\}$$

(2.5.15) For $B_1=0, B_2\neq 0$ and $q=0$, the attained result (2.5.15) becomes

$$\psi(x,t)=e^{i\eta}\left\{\sqrt{\delta}\left(\frac{a+3kc-2k\rho d-\rho b-\omega d}{2(k\theta+k\lambda)}\right)\tan(\sqrt{\delta}\,\xi)\right.$$

$$\left.+\frac{\delta\sqrt{S}\sqrt{a+3kc-2k\rho d-\rho b-\omega d}}{B_2\sqrt{2k\delta\theta+2k\delta\lambda}}\sec(\sqrt{\delta}\,\xi)\right\}.$$

(2.5.16) Again, putting $B_1\neq 0, B_2=0$ and $q=0$, solution (2.5.15) yields

$$\psi(x,t)=e^{i\eta}\left\{\sqrt{\delta}\left(\frac{a+3kc-2k\rho d-\rho b-\omega d}{2(k\theta+k\lambda)}\right)\cot(\sqrt{\delta}\,\xi)\right.$$

$$\left.+\frac{\delta\sqrt{S}\sqrt{a+3kc-2k\rho d-\rho b-\omega d}}{B_1\sqrt{2k\delta\theta+2k\delta\lambda}}\operatorname{cosec}(\sqrt{\delta}\,\xi)\right\}.$$

(2.5.17) Set 2: $a_0=0, a_1=-\sqrt{\frac{a+3kc-2k\rho d-\rho b-\omega d}{2(k\theta+k\lambda)}}, b_1=\frac{-i\sqrt{q^2-\delta^2 S}\sqrt{a+3kc-2k\rho d-\rho b-\omega d}}{\sqrt{2k\delta\theta+2k\delta\lambda}}$

$$\omega=\frac{1}{2}(-2ak^2-2ck^3+2k\omega b+a\delta+3kc\delta-2k\rho d\delta-\delta\rho b+2k^2\omega d-\delta\omega d).$$

By inserting the estimations gathered above into Eq. (2.5.5), we ascertain

$$U(\xi)=-\sqrt{\frac{a+3kc-2k\rho d-\rho b-\omega d}{2(k\theta+k\lambda)}}\,\frac{i\sqrt{\delta}\left\{B_1\cos(\sqrt{\delta}\,\xi)-B_2\sin(\sqrt{\delta}\,\xi)\right\}}{B_1\sin(\sqrt{\delta}\,\xi)+B_2\cos(\sqrt{\delta}\,\xi)+\frac{q}{\delta}}$$

$$\frac{+i\sqrt{q^2-\delta^2 S}\sqrt{a+3kc-2k\rho d-\rho b-\omega d}}{\sqrt{2k\delta\theta+2k\delta\lambda}\left\{B_1\sin(\sqrt{\delta}\,\xi)+B_2\cos(\sqrt{\delta}\,\xi)+\frac{q}{\delta}\right\}}$$

(2.5.18) together with $\xi=\frac{1}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}-\frac{\rho}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \eta=\frac{-k}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{\omega}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}$. Thus, inserting (2.5.18), we attain exact soliton solution of Eq. (2.5.1) in the form:

$$\psi(x,t)=e^{i\eta}\left\{-\sqrt{\frac{a+3kc-2k\rho d-\rho b-\omega d}{2(k\theta+k\lambda)}}\,\frac{i\sqrt{\delta}\left\{B_1\cos(\sqrt{\delta}\,\xi)-B_2\sin(\sqrt{\delta}\,\xi)\right\}}{B_1\sin(\sqrt{\delta}\,\xi)+B_2\cos(\sqrt{\delta}\,\xi)+\frac{q}{\delta}}\right.$$

$$\left.+\frac{i\sqrt{q^2-\delta^2 S}\sqrt{a+3kc-2k\rho d-\rho b-\omega d}}{\sqrt{2k\delta\theta+2k\delta\lambda}\left\{B_1\sin(\sqrt{\delta}\,\xi)+B_2\cos(\sqrt{\delta}\,\xi)+\frac{q}{\delta}\right\}}\right\}$$

(2.5.19) If $B_1=0, B_2\neq 0$ and $q=0$, the required solution of Eq. (2.5.19) is

$$\psi(x,t)=e^{i\eta}\left\{\sqrt{\delta}\left(\frac{a+3kc-2k\rho d-\rho b-\omega d}{2(k\theta+k\lambda)}\right)\tan(\sqrt{\delta}\,\xi)\right.$$

$$+\frac{\delta\sqrt{S}\sqrt{a+3kc-2k\rho d-\rho b-\omega d}}{B_2\sqrt{2k\delta\theta+2k\delta\lambda}}\sec\left(\sqrt{\delta}\,\xi\right)\Bigg\}.$$

(2.5.20) If we take, $B_1\neq 0, B_2=0$ and $q=0$, solution (2.5.19) will be altered to

$$\psi(x,t)=e^{i\eta}\Bigg\{-\sqrt{\delta}\left(\frac{a+3kc-2k\rho d-\rho b-\omega d}{2(k\theta+k\lambda)}\right)\cot\left(\sqrt{\delta}\,\xi\right)$$

$$+\frac{\delta\sqrt{S}\sqrt{a+3kc-2k\rho d-\rho b-\omega d}}{B_1\sqrt{2k\delta\theta+2k\delta\lambda}}\operatorname{cosec}\left(\sqrt{\delta}\,\xi\right)\Bigg\}.$$

(2.5.21)

Case 3: When ($\delta=0$), in this case, we get a set of solution, which is presented below:

$$a_0=0, a_1=a_1, b_1=b_1, \omega=\frac{k\left(k(a+ck)+(\theta+\lambda)a_0\right)}{-1+bk+dk^2}, k=\frac{-b+\sqrt{b^2+4d}}{2d}$$

Putting these values in Eq. (2.5.5), we get

$$U(\xi)=\frac{2a_1(B_1+\xi)}{\xi^2+2B_1\xi+2B_2}+\frac{2b_1}{\xi^2+2B_1\xi+2B_2},$$

(2.5.22) with $\xi=\frac{1}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}-\frac{\rho}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \eta=\frac{-k}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{\omega}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}$. Hence, the desired solution of Eq. (2.5.1) is derived from (2.5.2) as follows:

$$\psi(x,t)=e^{i\eta}\left(\frac{2a_1(B_1+\xi)}{\xi^2+2B_1\xi+2B_2}+\frac{2b_1}{\xi^2+2B_1\xi+2B_2}\right).$$

(2.5.23) If we take $B_1=0$ and $B_2\neq 0$, solution (2.5.23) will be altered to

$$\psi(x,t)=e^{i\eta}\left(\frac{2a_1\xi}{\xi^2+2B_2}+\frac{2b_1}{\xi^2+2B_1\xi}\right).$$

(2.5.24) Similarly, for $B_1\neq 0$ and $B_2=0$, the solution (2.5.23) takes the form

$$\psi(x,t)=e^{i\eta}\left(\frac{2a_1(B_1+\xi)}{\xi^2+2B_1\xi}+\frac{2b_1}{\xi^2+2B_1\xi}\right).$$

(2.5.25)

3. Results Comparison by The Double $(\phi'/\phi, 1/\phi)$-expansion Method and The Hirota Trilinear Method

3.1. Introduction and Background

The exact analytical solutions to some NLEEs with fractional order are constructed some of which are fresh and further generic than the documented results in the literature. This chapter illustrates

the achieved solutions through depicting the graphs by the computational software Mathematica. The graphs include different shapes of soliton like-asymptotic, anti-asymptotic, periodic, smooth periodic, plane type, parabolic, breathing, irregular periodic, breathing-like periodic, rhythmic-breather, positive periodic shape, V-shaped, homoclinic-breather, heteroclinic-breather, multi-soliton shape, lump-periodic, breather-lump periodic, breather-periodic and general soliton solutions. A graph is a powerful tool for communicating mathematical ideas. It can provide a concise and understandable explanation of any state. This section provides two- and three-dimensional diagrams of some of the earlier identified solutions to help explain the phenomenon. It is revealed through three- and two-dimensional graphs that the characteristics of the portrayals are changed by varying the fractional-order value while retaining the values of all the other parameters. For the sake of brevity and to avoid the repetition of analogous solitons, other derived solutions to the equations generate identical solitons for varying values of the free parameters, which are not displayed. It is established from the discussion that the solitons change depending on the values of fractional-order and shown in the soliton profiles in two- and three-dimensional diagrams. It is noticed from the soliton profiles that each solution form fractional and integer values exhibit significant variation in the two- and three-dimensional sketches. We have exposed the evolution of the solitons through 2D and 3D graphs by changing the values of fractional-order while keeping the values of the other parameters fixed. The solutions of the fractional perturbed nonlinear Schrödinger Kerr law, nonlinear Schrödinger unstable, Kaup-Newell, and Biswas-Arshed equations are explained. Their substantive layout is studied for varying the controlling parameter values using the Mathematica program. We described the 3D plots and associated 2D graphs, and the contour plots have also been depicted for better comprehension. We discuss the graphical representations of the solutions obtained for the Nizhnik-Novikov-Veselov, Schrödinger Kerr law, nonlinear Schrödinger unstable, Schrödinger-Hirota, Biswas-Arshed, Kaup-Newell equations using the two-variable $(\phi'/\phi, 1/\phi)$-expansion technique for various parameters estimations. We have categorized the solutions into three parts-modulus, real, and imaginary components-to enhance comprehension and demonstrate the physical significance of the achieved solutions. Through 3D, 2D, and Contour graphs, we reveal that the characteristics of these representations change as the fractional-order value varies while keeping the values of all parameters constant.

In optical fibers, fluid dynamics, plasma physics, surface wave propagation, and many other domains, exact and soliton-type solutions to fractional nonlinear evolution equations arising in nonlinear sciences are of critical importance. This is why scientists have focused extensively on finding solutions using various approaches to achieving better outcomes and understanding a wide range of complex phenomena relevant to everyday life. In this chapter, the obtained results are compared with those found in the literature.

3.2. A comparison table to the fractional (2+1)-dimensional Nizhnik-Novikov-Veselov equation

The time fractional (2+1)-dimensional Nizhnik-Novikov-Veselov equation was investigated and constructed three solutions by making use of the Hirota trilinear method (Wang, and Dai, 2015). On the other hand, we have implemented the double $(\phi'/\phi, 1/\phi)$-expansion method to the same equation and achieve eighteen solutions. The obtained solutions are compared with the existing results through the following table:

Obtained solutions	Wang, and Dai, (2015)
$\psi(x,y,t)=\frac{k}{\delta}\delta-\delta\tanh^2(\sqrt{-\delta}\times(x+$	$\psi(x,y,t)=\psi_0-\frac{4(b_1+b_2)}{(x+b_1y-Nt)^2+(x+b_2y+Ct)^2+M}+8($
$\psi(x,y,t)=\delta-\delta\coth^2(\sqrt{-\delta}(x+y-$	$\psi(x,y,t)=\frac{x((b_1+b_2)x+(b_1^2+b_2^2)y-(Nb_1-Cb_2)t}{((x+b_1y-Nt)^2+(x+b_2y+Ct)^2+M)^2}$
$\psi(x,y,t)=\frac{3}{2}\delta+\delta\tan^2(\sqrt{\delta}(x+y-\frac{k}{\delta}$	$\psi(x,y,t)=\frac{8}{(x+b_1y-Nt)^2+(x+b_2y+Ct)^2+M}+\frac{8((2}{((x+b_1$
$\psi(x,y,t)=a_0+\frac{b_1}{B_2}$	$\psi(x,y,t)=\frac{-4(b_1^2+b_2^2)}{(x+b_1y-Nt)^2+(x+b_2y+Ct)^2+M}+\frac{8((b_1+}{((x+b$

3.3. A comparison table to the space-time fractional nonlinear Schrödinger Kerr law equation

The space-time fractional nonlinear Schrödinger Kerr law equation is demonstrated via the generalized tanh method and attained only six solutions (Inc et al., 2018), where as the double $(\phi'/\phi,1/\phi)$-expansion method has provided nineteen solutions. For simplicity, we have recorded only six solutions in this study. The following table brings up the comparison of the solutions:

Obtained solutions	Inc et al. (2018)
$\psi(x,t)=e^{i\eta}\{\sqrt{-\delta}\sqrt{\frac{-bk^2+k^2v}{\sqrt{2}c}}\tanh(\sqrt{-\delta}\xi$	$\psi(x,t)=(-\sqrt{\frac{2(-bk^2+ak^2v)}{c}}\sqrt{-C}\tanh(\sqrt{-C}\xi))$
$\psi(x,t)=e^{i\eta}\{\sqrt{-\delta}\sqrt{\frac{-bk^2+k^2v}{\sqrt{2}c}}\coth(\sqrt{-\delta}\xi$	$\psi(x,t)=(-\sqrt{\frac{2(-bk^2+ak^2v)}{c}}\sqrt{-C}\coth(\sqrt{-C}\xi))$
$\psi(x,t)=e^{i\eta}\{\sqrt{\frac{k\sqrt{\delta}s(v-b)}{\sqrt{2}c}}\tan(\sqrt{\delta}\xi)-\frac{k\sqrt{}}{}$	$\psi(x,t)=(\sqrt{\frac{2(-bk^2+ak^2v)}{c}}\sqrt{C}\tan(\sqrt{C}\xi))\times e^{i(kx^\beta}$
$\psi(x,t)=e^{i\eta}\{\sqrt{\frac{k\sqrt{\delta}s(v-b)}{\sqrt{2}c}}\cot(\sqrt{\delta}\xi)-\frac{k\sqrt{}}{}$	$\psi(x,t)=(-\sqrt{\frac{2(-bk^2+ak^2v)}{c}}\sqrt{C}\cot(\sqrt{C}\xi))\times e^{i(k}$
$\psi(x,t)=e^{i\eta}(\frac{2a_1\xi}{\xi^2+2B_1\xi})$	$\psi(x,t)=(-\sqrt{\frac{2(-bk^2+ak^2v)}{c}}\sqrt{\frac{\omega}{2Kv}}\tanh(\sqrt{\frac{\omega}{2Kv}}\xi$
$\psi(x,t)=e^{i\eta}(\frac{2a_1(B_1+\xi)}{\xi^2+2B_1\xi})$	$\psi(x,t)=(-\sqrt{\frac{2(-bk^2+ak^2v)}{c}}\sqrt{\frac{\omega}{2Kv}}\coth(\sqrt{\frac{\omega}{2Kv}}\xi$

3.4. A comparison table to the space-time fractional unstable nonlinear Schrödinger equation

Tebue et al. (2016) studied the space-time fractional unstable nonlinear Schrödinger equation by the exponential rational function method and obtained two solutions only. We have implemented the double $(\phi'/\phi,1/\phi)$-expansion method which provides twenty solutions. A table to compare the gained solutions and existing solutions is given below.

Obtained solutions	Tebue et al. (2016)
$\psi(x,t)=e^{i\eta}\{\sqrt{\frac{\sqrt{\delta}}{2\sqrt{\lambda}}}\cot$	$\psi(x,t)=e^{i(\beta x+rt)}[\frac{1}{2}\sqrt{\frac{2\beta^2+4\gamma+2r}{\lambda}}\pm\sqrt{\frac{2\beta^2+4\gamma+2r}{\lambda}}\frac{}{1+\cosh(\sqrt{-2\beta^2-4}$
$\psi(x,t)=e^{i\eta}\{\sqrt{\frac{\sqrt{\delta}}{2\sqrt{\lambda}}}\tan$	$\psi(x,t)=e^{i(\beta x+rt)}[\frac{1}{2}\sqrt{\frac{2\beta^2+4\gamma+2r}{\lambda}}\pm\sqrt{\frac{2\beta^2+4\gamma+2r}{\lambda}}\frac{}{1+\cos(\sqrt{2\beta^2+4\gamma+}$

3.5. A comparison table to the space-time fractional Schrödinger-Hirota equation

The space-time fractional Schrödinger-Hirota equation was studied by Ekici et al. (2017) using the trial equation method and constructed thirteen solutions and we have achieve nineteen solutions by using the double $(\phi'/\phi,1/\phi)$-expansion method. A few obtained solutions are given in the following table to compare with those found by Ekici et al. (2017):

Obtained solutions	Ekici et al. (2017)
$\psi(x,t)=e^{i\eta}\{a_1\sqrt{-\delta}\tanh(\sqrt{-}$	$\psi(x,t)=\sqrt{\tau_0+\tau_1\lambda_1\pm\frac{\tau_1 W}{1+b\omega\kappa}}\sqrt{x+\frac{a+2a\kappa+b\omega^2}{1+b\omega\kappa}t-\xi_0}\times\exp[i\{-\kappa$
$\psi(x,t)=e^{i\eta}\{a_0+a_1\sqrt{-\delta}\coth($	$\psi(x,t)=\pm\sqrt{\frac{\tau_1 W}{1+b\omega\kappa}}\sqrt{x+\frac{a+2a\kappa+b\omega^2}{1+b\omega\kappa}t-\xi_0}\exp[i\{-\kappa x+\frac{a\mu_1\tau_1-}$
$\psi(x,t)=e^{i\eta}\{a_0-a_1\sqrt{\delta}\tan(\sqrt{($	$\psi(x,t)=\sqrt{\frac{\lambda_2-\lambda_1}{2}}\sqrt{\tau_1\{1\mp\coth[\frac{\lambda_1-\lambda_2}{2W}(x+\frac{a+2a\kappa+b\omega^2}{1+b\omega\kappa}t)]\}}\times e$

3.6. Comparison table to the space-time fractional Biswas-Arshed equation

Yildirim (2019) applied the trial equation technique to examine the space-time fractional Biswas-Arshed equation and attained sixteen solutions only. We have implemented the double $(\phi'/\phi,1/\phi)$-expansion method which provided twenty solutions. A few of the solutions obtained, which are compared with the findings of Yildirim (2019), are shown in the table below:

Solutions obtained by the double $(\phi'/\phi,1/\phi)$-expansion method	Solutions obtained by Yildirim (2019)
$\psi(x,t)=e^{i\eta}\{\sqrt{-\delta}\sqrt{\frac{a+3kc-2kpd-pb-\omega d}{\sqrt{2}(\kappa\theta+\kappa\lambda)}}\tanh$	$\psi(x,t)=\pm\sqrt{\frac{2(\omega+\kappa^2\alpha_1-\kappa\omega\alpha_2-\kappa^2\omega b_2+k^3b$
$\psi(x,t)=e^{i\eta}\{\sqrt{-\delta}\sqrt{\frac{a+3kc-2kpd-pb-\omega d}{\sqrt{2}(\kappa\theta+\kappa\lambda)}}\coth$	$\psi(x,t)=\pm\sqrt{\frac{2(\omega+\kappa^2\alpha_1-\kappa\omega\alpha_2-\kappa^2\omega b_2+k^3b$
$\psi(x,t)=e^{i\eta}\{\sqrt{\delta}\sqrt{\frac{a+3kc-2kpd-pb-\omega d}{\sqrt{2}(\kappa\theta+\kappa\lambda)}}\tan(\sqrt{($	$\psi(x,t)=\pm\sqrt{\frac{\omega+\kappa^2\alpha_1-\kappa\omega\alpha_2-\kappa^2\omega b_2+k^3b_1}{\kappa(\lambda+\theta)}}$
$\psi(x,t)=e^{i\eta}\{\sqrt{\delta}\sqrt{\frac{a+3kc-2kpd-pb-\omega d}{\sqrt{2}(\kappa\theta+\kappa\lambda)}}\cot(\sqrt{($	$\psi(x,t)=\pm\sqrt{\frac{\omega+\kappa^2\alpha_1-\kappa\omega\alpha_2-\kappa^2\omega b_2+k^3b_1}{\kappa(\lambda+\theta)}}$

Solutions obtained by the double $(\phi'/\phi, 1/\phi)$-expansion method	Solutions obtained by Yildirim (2019)
$\psi(x,t)=e^{i\eta}\left(\frac{2a_1(B_1+\xi)}{\xi^2+2B_1\xi+2B_2}+\frac{2b_1}{\xi^2+2B_1\xi+2B_2}\right)$	$\psi(x,t)=\pm\sqrt{\frac{\omega+\kappa^2\alpha_1-\kappa\omega\alpha_2-\kappa^2\omega b_2+k^3b_1}{\kappa(\lambda+\theta)}}$
$\psi(x,t)=e^{i\eta}\left(\frac{2a_1(B_1+\xi)}{\xi^2+2B_1\xi}+\frac{2b_1}{\xi^2+2B_2}\right)$	$\psi(x,t)=\pm\sqrt{\frac{\omega+\kappa^2\alpha_1-\kappa\omega\alpha_2-\kappa^2\omega b_2+k^3b_1}{\kappa(\lambda+\theta)}}$

4. Review of Results and Next Steps

4.1. Review of Results

This study demonstrates the profound impact of fractional derivatives on optical fiber and ion-acoustic models. Fractional derivatives, especially of non-integer orders, provide flexibility in describing the temporal and spatial evolution of waveforms in these models which offer potential avenues for improving signal transmission in optical fibers and refining theoretical modeling of ion-acoustic waves in plasmas. Optical fiber and ion-acoustic models, as a whole, serve as powerful tools for enhancing the accuracy and adaptability of models using fractional derivatives. In this dissertation, we have implemented an analytical approach for finding traveling wave solutions to the FNLEEs that appear in mathematical physics and applied mathematics, using the beta derivative in the context of the double $(\phi'/\phi, 1/\phi)$-expansion method. Using these approaches, we have successfully derived closed-form traveling wave solutions for the (2+1)-dimensional time-fractional Nizhnik-Novikov-Veselov equation, the space-time fractional nonlinear Schrödinger Kerr law equation, the space-time fractional unstable nonlinear Schrödinger equation, the space-time fractional Schrödinger-Hirota equation, the space-time fractional Biswas-Arshed equation, the space-time fractional Sasa-Satsuma equation, the space-time fractional nonlinear Kaup-Newell equation, and the (2+1)-dimensional extended time-fractional Zakharov-Kuznetsov equation. We have obtained solitary wave solutions for all the analyzed equations with specific free parameters using the double $(\phi'/\phi, 1/\phi)$-expansion technique. For certain values of these free parameters, the traveling wave solutions yield various types of solitary wave solutions. These solutions, constructed using rational, trigonometric, and hyperbolic functions, provide insight into the complex nonlinear physical processes that occur in the natural world. The solutions include V-shaped, positive-periodic, anti-asymptotic, breather, kink-bell-shaped, general solutions, asymptotic, periodic, kink, bell-shaped, anti-kink, periodic-type solitons, and others. We have obtained solitary wave solutions for all the analyzed equations with specific free parameters using the double $(\phi'/\phi, 1/\phi)$-expansion technique. For certain values of these free parameters, the traveling wave solutions yield various types of solitary wave solutions. Therefore, this powerful, effective, and straightforward method can also be used to investigate other kinds of FNLEEs that frequently arise in various real-world scientific applications.

4.2. Next Steps

Obtained closed-form traveling wave solutions for several FNLEEs using the double $(\phi'/\phi, 1/\phi)$-expansion method with the beta derivative. The graphical representation has illustrated how the

order of the fractional derivatives varies depending on the medium of the solutions obtained. It can aid in a better understanding of how various real-world phenomena are simulated using fractional derivatives. That's why we believe that our work is an important step towards achieving closed-form solutions to various physical problems dedicated to FNLEEs. Interdisciplinary applications of FNLEE can be explored in areas such as signal processing, optical communication systems, fractional dynamics, medical imaging, and control theory, plasma physics, where fractional dynamics and nonlinear behavior play an important role. This research can further improve the understanding and application of fractional- order models in various fields of science and engineering in future. Machine learning techniques can be applied to analyze and predict the solution of FNLEEs. For example, neural networks can be used to approximate solutions or identify patterns in solution behavior across different parameter spaces. This could provide practical relevance and open new avenues for research.

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